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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

CONSTRUCTION OF PLANE CURVES OF GIVEN ORDER AND GENUS, HAVING DISTINCT DOUBLE POINTS.

BY PROFESSOR VIRGIL SNYDER.

(Read before the American Mathematical Society, September 10, 1908.)

In researches in the theory of birational transformations it is frequently desirable to employ curves of given order and given genus, all of whose singularities are ordinary distinct double points; but the possibility of finding such curves has been assumed. In the following note I show that such curves exist for every value of the genus p not exceeding $\frac{1}{2}(n-1)(n-2)$, n being the order of the curve, and determine the equation in each case.

1. Points on the quadric surface $F \equiv xz - yw = 0$ may be defined by simultaneous values of $x_1 : x_2$ and $y_1 : y_2$, where

$$\frac{x}{y} = \frac{w}{z} = \frac{x_1}{x_2}, \quad \frac{x}{w} = \frac{y}{z} = \frac{y_1}{y_2}.$$

An algebraic curve lying on F_2, cutting the generators of one system in r points, and those of the other in $n-r$ points may be expressed by an equation of the form

$$f\left(\overset{r}{\frac{x_1}{x_2}}, \overset{n-r}{\frac{y_1}{y_2}}\right) = 0 \qquad (r \leqq n-r).$$

By multiplying this equation by a suitable power of x_1 and making use of the relations

$$x = x_1y_2, \quad y = x_2y_1, \quad z = x_2y_2, \quad w = x_1y_1,$$

XV 1908

the equation $f=0$ may
F_{n-r} containing our sp
section with F_2 consists
If now between $F_2=0$,
the resulting equation
vertex at (0, 1, 0, 0) o
by the plane $y=0$ is a
0, 1, 0) and an $(n-r)$-
singularities, when the
the correspondence has

If y be now transf
remaining fixed, the eq
but F_2 becomes $w^2 + y$
coordinates (x, y, z, w)
(ξ, ζ, ω) in $y = 0$ are fo

$$\sigma x = \xi\omega, \quad \sigma y =$$

It is an ordinary ster
rational.

Again, if between F
eliminated, the resultin
cone from (0, 0, 0, 1)
Since the vertex is not
generators can cut F_2 in
of K'_n can contain no hi

This idea can now be
curves, without any re
By superposing the tw
way that $\eta' = \omega$, the wh
plane. The result can

Given the plane cur
at (1, 0, 0) *and an* $(n -$
point singularities. It
another curve of the sam
points by means of the t

$$\rho x' = xz,$$

In particular, if $r =$
every order exist, having
homogeneous form, the

transformed into another rational curve of the same order by eliminating x, y from the equations

$$y = xf_{n-1}(x), \quad x'y = x, \quad y y = xy - 1,$$

the new curve in x', y' having distinct double points. For $r = 2$, hyperelliptic curves of genus $n - 3$, i. e., having

$$h = \tfrac{1}{2}(n - 2)(n - 3) + 1$$

double points, result. To obtain similar curves of lower genera, it is necessary to impose double coincidences in the correspondence $f(x_1 : x_2, y_1 : y_2) = 0$. A more general method for obtaining curves which are not hyperelliptic and whose genus always lies in the same interval can be at once derived from the theorem of Halphen * that every plane curve of order n having at least h double points can be obtained by projecting some non-singular space curve on the plane. It is only necessary that the center of perspective be not on the ruled surface of trisecants of the space curve.

The method of Halphen † can also be applied in case $h \geqq \frac{1}{3}(n - 1)(n - 2)$, but for lower values down to the largest integer contained in $[\frac{1}{2}(n - 1)]^2$ this is no longer possible. But for this interval the number of conditions to be satisfied is sufficiently small to insure that the required number of actual double points can be imposed on the space curve without making it composite. The number of double points is less than $\frac{1}{12}(n - 5)(n - 1)$, while the number of constants still at our disposal is at least $\frac{1}{2}(n + 2)^2$.

2. For values of p larger than $\frac{1}{4}(n - 1)(n - 3)$ the preceding method will not apply, as no space curve can have a larger genus, but in this case that of Jonquières ‡ can be readily extended to define the desired curves. Let c_l, c'_l, c''_l be three curves belonging to a net having m base points in common. The curve

$$c_l c''_l \phi_{n-2l} - c'^2_l \psi_{n-2l} = 0$$

will then have these m base points for double points and the constants in ϕ_{n-2l}, ψ_{n-2l} can be so chosen that it will have no others.

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* "Sur la classification des courbes gauches algébriques," *Jour. de l'Ecole polytech.*, vol. 52 (1882), pp. 1-200. See page 27.

† *l. c.*, p. 126.

‡ *Mém. de l'Acad. de Paris*, vol. 16 (1858).

The maximum value of m is $l^2 - l + 1$,* and the largest value of l is the largest integer in $\frac{1}{2}n$. When n is even, say $2l$, these curves are already included among those treated before for $r = \frac{1}{2}n - 1$. By taking smaller values for l, curves of all the higher genera can be constructed.

3. When n is odd, say $n = 2l + 1$, the lowest genus obtainable directly is $l^2 - 1$, while the highest one from the space curve is $l^2 - l$. By passing a pencil of c_{l+1} through the l^2 base points of a pencil of c_l, the lower genus can be obtained, and the higher one by projection. To obtain the intermediate cases, pass two c_l through $l - 1$ points on a fixed straight line c_1. Through κ of the $l^2 - l + 1$ remaining points of intersection, the same $l - 1$ points on c_1, and two other points on c_1 pass a pencil of c_{l+2}. The two pencils are now to be made projective in such a way that corresponding curves intersect on c_1. The locus of the point of intersection will be a c_{2l+2}, but c_1 will be a factor. The remaining c_{2l+1} will have $\kappa + l - 1$ double points, and no other singularities. κ can have any value from 0 to $l^2 - l + 1$. This completes the solution of the problem.†

CORNELL UNIVERSITY,
July, 1908.

ON PERIODIC LINEAR SUBSTITUTIONS WHOSE COEFFICIENTS ARE INTEGERS.

BY DR. ARTHUR RANUM.

(Read before the American Mathematical Society, September 6, 1907.)

1. THE object of this note is to call attention to one or two theorems that follow easily from the results of my paper in this BULLETIN, April, 1907, volume 13, pages 336–345, by taking into account a theorem of Minkowski's given in *Crelle*,

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* C. Kupper: "Ueber das Vorkommen von linearen Schaaren ···" *Sitzungsbericht der Böhm. Gesell.*, Prag, 1892, pp. 264–262. In my article "On curves having a net of minimum adjoint curves," BULLETIN, vol. 14, page 70 (1907) I showed how such a net of curves can be actually constructed by rational operations.

† Since the basis points are not independent, Cayley's theorems regarding the configuration of residual points of intersection do not apply. See Küpper, "Bestimmung der Maximalbasis B für eine irreducible μ-fache Mannigfaltigkeit von Curven $n$ter Ordnung," *Monatshefte für Math. u. Physik*, vol. 6, (1895), pp. 5–11, and my own paper, "On birational transformations of curves of high genus," *Amer. Jour. Math.* vol. 30 (1908), pp. 10–20.

volume 100, pages 449–456, and volume 101, pages 196–202. They relate to linear homogeneous substitutions of finite period in n variables, whose coefficients are integers, and whose determinants are different from zero and are therefore ± 1.

2. Two such linear substitutions (matrices) are said to be congruent mod k, if their corresponding coefficients (elements) are congruent mod k. Letting I stand for the identical matrix, Minkowski's theorem may be stated as follows:

The only periodic matrix that is congruent to I mod k, if $k > 2$, is I itself, of period 1; while the finite periods of the matrices that are congruent to I mod 2, are 1 and 2.

As illustrations of this theorem, consider the matrix $\left(\begin{smallmatrix}1+3a & 3b\\ 3c & 1+3d\end{smallmatrix}\right)$, which is congruent to $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ mod 3, but is of infinite period unless $a = b = c = d = 0$; and on the other hand the matrix $\left(\begin{smallmatrix}1 & 0\\ 2c & -1\end{smallmatrix}\right)$, which is congruent to $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ mod 2, and is of period 2.

3. Let us now determine what happens to the period of a matrix when it is considered as belonging to a linear congruence group, or Jordan group, whose modulus is k. In that case any two matrices which are congruent mod k become identical, and a matrix that is congruent to I mod k becomes I itself. The period of every matrix becomes finite, whether it was originally finite or infinite; if it was originally finite ($= m$), it not only remains finite, but obviously becomes a divisor of m. The following theorem can now be readily proved:

If a matrix L whose elements are integers is of finite period m, then the corresponding matrix L', belonging to the Jordan group whose modulus is k, is also of period m, if $k > 2$; while if $k = 2$, its period is either m or $\frac{1}{2}m$.

Proof. Let the period of L' be m', and put $m/m' = r$, which is an integer. Then $(L')^{m'} = I$, and $L^{m'}$, which is clearly of period r, is at the same time congruent to I mod k. Therefore, by Minkowski's theorem, we see that if $k > 2$, $r = 1$ and $m' = m$; while if $k = 2$, $r = 1$ or 2, and $m' = m$ or $\frac{1}{2}m$.

As an illustration of this theorem, consider the matrix $\left(\begin{smallmatrix}0 & -1\\ 1 & 1\end{smallmatrix}\right)$, whose period, irrespective of any modulus, is 6; with respect to the modulus 3 its period remains 6, while with respect to the modulus 2 its period becomes 3.

4. This result furnishes a short cut to the following theorem, which was otherwise proved in my paper in the *Transactions*, April, 1908, volume 9, page 191, Corollary 2.

The highest prime period of an n-ary matrix whose elements are integers cannot be greater than $n + 1$.

Proof. Let L be an n-ary matrix of period p, a prime, and suppose that L' is the corresponding matrix of the Jordan group whose modulus is $p^a (a > 1)$; by the theorem of § 3, L' is also of period p. Now suppose that p could be $> n + 1$ (and therefore > 2); then, in view of theorems 2 and 5 of the BULLETIN paper referred to above (pages 339 and 342, respectively),* the matrix L'_1, being excluded from $H - H_1$ and $H_1 - H_2$ (in the notation of that paper), would necessarily belong to $H_{a-1} - I$, and would be congruent to $I \bmod p^{a-1}$. Therefore L would itself be congruent to $I \bmod p^{a-1}$. But by Minkowski's theorem this is impossible.

5. In a similar manner, by reference to theorems 3 and 4 of the same BULLETIN paper, it is easily seen that if $n \leqq p^a$, where p is an odd prime, then the period of an n-ary matrix whose elements are integers cannot be a power of p higher than the ath.†

6. The general problem of finding all the finite periods of n-ary matrices whose elements are integers can be solved at once by observing that they are identically the same as the finite periods of n-ary matrices whose elements are rational numbers. In other words, the use of rational fractions as the elements of a matrix does not make possible any finite periods not already existing in the case of matrices whose elements are integers.

This follows from the fact (*Transactions*, l. c., §§ 21–27) that for every matrix of finite period with rational elements there exists a canonical form, of the same period, whose elements are integers.

Accordingly, the results given in §§ 10–20 of the paper just referred to furnish a complete solution of the problem of this section.

CORNELL UNIVERSITY,
June, 1908.

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* Minkowski's theorem itself can also be derived from the results given in that paper.

† It is to be noticed, however, that this result is not so strong or complete as the corresponding one given in the *Transactions*, l. c., p. 191, Corollary 2.

EVEN MULTIPLY PERFECT NUMBERS OF FIVE DIFFERENT PRIME FACTORS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, April 27, 1907.)

THE problem of the existence of multiply perfect numbers has been completely solved for the cases in which the numbers contain not more than four different prime factors.* The object of this note is to give the results of a similar study of even multiply perfect numbers of five different prime factors. It will be found that there are three such numbers of multiplicity 4 and one of multiplicity 3.

The lemma † which formed the starting point of a previous paper will be repeated:

LEMMA. *If x is a positive integer > 1, $x^t - 1$ has a prime factor not dividing $x^u - 1$ $(u < t)$, except in the cases $t = 2$, $x = 2^v - 1$, $v \geqq 2$; $t = 6$, $x = 2$. Such prime factors of $x^t - 1$ are of the form $st + 1$, and evidently if t is odd and > 1, they are of the form $2st + 1$.*

Such a prime, $st + 1$ or $2st + 1$, is called a characteristic factor.

Throughout we shall represent the numbers by

$$N = 2^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} p_4^{\alpha_4} p_5^{\alpha_5}, \tag{1}$$

where $p_2 < p_3 < p_4 < p_5$ are different odd primes. If further we denote the multiplicity by m, we have readily the following fundamental formulas:

$$m = \frac{2^{\alpha_1+1} - 1}{2^{\alpha_1}} \cdot \frac{p_2^{\alpha_2+1} - 1}{p_2^{\alpha_2}(p_2 - 1)} \cdots \frac{p_5^{\alpha_5+1} - 1}{p_5^{\alpha_5}(p_5 - 1)}, \tag{2}$$

$$m < 2 \cdot \frac{p_2}{p_2 - 1} \cdots \frac{p_5}{p_5 - 1}. \tag{3}$$

From (3) it follows that $m \leqq 4$, as may be seen by substituting 3, 5, 7, 11 for the p's. But $m \neq 2$; for all even numbers of multiplicity 2 have only two prime factors.* Hence we have to consider the two cases $m = 4$, $m = 3$.

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* Carmichael, *Annals of Mathematics*, ser. 2, vol. 8, pp. 49–56, 149–158.

† Carmichael, *l. c.*, p. 151.

When $m = 4$ it is easy by inequality (3) to show that $p_2 = 3$ and $p_3 = 5$ or 7 and also to restrict the values of p_4 and p_5. Then by the help of the lemma applied to expressions of the form $p^{a+1} - 1$ the values of the exponents a in (1) may be restricted. Then by an examination of the remaining possibilities I have found that equation (2) is never possible when $p_3 = 7$. On the other hand, when $p_3 = 5$, there are three cases when (2) is satisfied; and consequently three multiply perfect numbers of multiplicity 4 as follows: $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, $2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31$, $2^7\ 3^3 \cdot 5^2 \cdot 17 \cdot 31$.

For $m = 3$ a similar discussion restricts p_2 to have the value 3 or 5. When $p_2 = 5$ equation (2) is impossible, as is shown again by examining the possibilities for p_3, p_4, p_5 and the exponents. When $p_2 = 3$ there is only one set of values satisfying (2), and these give a perfect number $2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127$, of multiplicity 3.

The results arrived at may be stated thus:

The even multiply perfect numbers of only five different prime factors are the following: $2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$, $2^9 \cdot 3^3 \cdot 5 \cdot 11 \cdot 31$, $2^7 \cdot 3^3 \cdot 5^2 \cdot 17 \cdot 31$ *of multiplicity* 4; *and* $2^{13} \cdot 3 \cdot 11 \cdot 43 \cdot 127$ *of multiplicity* 3.

All these numbers are included in the table published in the BULLETIN, volume 13, pages 385–386. A misprint occurs in the table: 23569921 for 23569920.

ALABAMA PRESBYTERIAN COLLEGE,
ANNISTON, ALA.

THE FOURTH INTERNATIONAL CONGRESS OF MATHEMATICIANS: SECTIONAL MEETINGS.

THE several sections of the Rome Congress had their first meetings on Tuesday morning, April 7. For each section one or more chairmen had been appointed to open the session, but at each the presiding officer was elected for the following day. In general those who opened the sessions merely welcomed the members and asked for nominations for president for that day. In each section a secretary was appointed who kept a full record of the proceedings. After the presentation of each paper, the secretary of the section asked the author to write a brief abstract. Through the kindness of the general secretary, Professor Castelnuovo, the reporter was furnished a copy of these abstracts, a liberal translation of which is given below. In general no abstracts were obtained for the papers presented by title; two others are also missing, viz., Professor Lamb's and Professor Capelli's. All the abstracts were to have been published in the "Bulletino" which was issued each morning, but owing to the difficulty of preparing them properly in so short a time this plan was not carried out. All papers and reports will be published in the Atti del Congresso, which is being prepared as rapidly as possible.

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* Carmichael, *l. c.*, p. 150.

SECTION I. ANALYSIS.

The section was opened by Professor Arzelà; the following acted as chairmen at different meetings: Jordan, Gordan, Forsyth, Pascal, E. H. Moore. The following papers were presented (to avoid repetition the abstracts immediately follow the titles):

GORDAN (Erlangen): "Die Auflösung der allgemeinen Gleichung 6ten Grades."

As principal result, the author proves that the equation of Professor Gerbaldi is the normal form of the equation of the sixth degree.

ZERMELO (Göttingen): "Über die Grundlagen der Arithmetik und Analysis."

From a new definition of infinite set are deduced the fundamental properties of the finite numbers, and in particular the principle of complete induction.

BOREL (Paris): "Sur les principes de la théorie des ensembles."

The author concludes that our exact notions of the power of a set do not come from anything else than the observations: there are enumerable sets and non-enumerable sets, this last definition being purely negative.

RIESZ (Budapest): "Stetigkeitsbegriff und abstrakte Mengenlehre."

For the consideration of the so-called properties of continuity of distinct and concrete varieties the author introduces the concepts of type of condensation and type of concatenation. These concepts form the foundations of a general theory of continuous varieties. Some problems of such a theory were also indicated.

FRIZELL (Cambridge, Mass.): "Die Mächtigkeit des Kontinuums."

First there is taken from the continuum a set consisting of those numbers in whose development in partial fractions every natural number appears only once. Then those sets of numbers were considered which are obtained in an analogous manner, admitting that single digits may be repeated or omitted. The sets in which an infinite number of digits are wanting are obtained as limiting cases. Finally the order of the continuum was determined.

KOEBE (Göttingen): "Über ein allgemeines Uniformisierungsprinzip."

After some historic observations, the author passed to the consideration of his most recent articles published in the *Göttinger Nachrichten* on the uniformization of algebraic or analytic curves by means of automorphic functions; he considered then the recent memoir of Poincaré, "Sur l'uniformisation des fonctions analytiques" (*Acta Mathematica*, volume 31) in order to characterize the analogous general problems which present themselves in consideration of groups of imaginary substitutions, and their solution.

BOUTROUX (Paris): "Sur l'inversion des fonctions entières."

The author discussed the relation of the problem to the general problem of the uniformization of the different mechanisms of permutations presented by the inverse of the integral functions, and showed how the general solution of the problem presents itself.

PETROVITCH (Belgrade): "Une classe remarquable de séries entières."

The author seeks the necessary and sufficient condition in order that a Maclaurin series have the property that all its zeros are real, and that the same be true of any polynomial formed of any number of terms at the beginning of the series. He gave some general properties of such a series, and established the existence and showed the formation of such a series in which any coefficient reaches its greatest possible maximum value. He gave also some properties of the transcendental function formed by this series.

PINCHERLE (Bologna): "Alcune spigolature nel campo delle funzioni determinanti."

Having recalled the definition, the history and the properties of determinants, the author put the infinite order of the generator in relation with the singularity of the determinant, and made an application to Dirichlet's series and to the creation of a succession of operations corresponding to the succession of the known functions typical for the order of infinity.

YOUNG (Cambridge): "On some applications of semi-continuous functions."

The author gave an account of the application which he has made of semi-continuous functions, 1) to the conditions of continuity, 2) to the theory of integration, 3) to the theory of measure, 4) to the discussion of the distinction of right and left, 5) to the theory of uniform convergence and of divergence of series. He added an account of its most recent applications to the theory of oscillating series, in which the concept of point of inferior and supérior semi-continuity has necessarily the leading rôle.

LAURA PISATI (Rome): "Saggio di una teoria sintetica delle funzioni di variabile complessa."

Doctor Pisati died March 30, and her paper was presented by title by Professor Marcolongo; it will appear in the "Atti del Congresso."

HADAMARD (Paris): "Sur l'application d'une méthode du calcul des variations."

The method indicated by the author in the *Comptes Rendus* (December, 1906) to demonstrate the existence of solutions of a problem in the calculus of variations does not present any particular difficulties in the case of the isoperimetric problems or in the case in which the integral is taken in the parametric form.

SCHLESINGER (Kolozswar): "Sur quelques problèmes paramétriques de la théorie des équations différentielles linéaires."

The author spoke of some problems which one encounters in studying the relations between the fundamental substitutions and the parameters on which the coefficients depend in the theory of linear differential equations.

Rémoundos (Athens): "Sur les zéros des intégrales d'une classe d'équations différentielles."

The author made use of a method of elimination which he used in his dissertation (Sur les zéros d'une classe de fonctions transcendantes, Paris, 1906) in order to extend a classic theorem of Picard to multiform functions of a finite number n of branches. He established a theorem analogous to that of his dissertation and considered the integrals of a very extended class of differential equations. The integrals of these equations either satisfy the above theorem or possess another property inherent in the existence of more integrals common to the differential equation considered and to those which express the derivatives u', u'' in functions of u.

Pick (Prag): "Über die Differentialgleichung der hypergeometrischen Funktion."

The author developed in a new aspect the general theory of the hypergeometric differential equation, considering in a more extended manner the case of equations of lower orders and having three singular points.

Soltykow (Krakow): "Sur l'existence des intégrales complètes de S. Lie et le perfectionnement de la méthode de Jacobi dans la théorie des équations aux dérivées partielles."

The author gives the general form of the partial differential equation which admits the complete integral of Lie of any order whatever. He makes some improvements on the method of Jacobi-Mayer, showing how the modifications thus introduced do not destroy the symmetry of the calculations of Jacobi. Every differential system of the characteristics takes the form of a canonical system of total differential equations.

Lalesco (Bucarest): "Sur les solutions analytiques de l'équation $\partial^2 Z/\partial x^2 = \partial Z/\partial y$."

It was shown that the solutions of the equation indicated which are analytic in a point are integral functions in Z of order at most equal to two, and that this property is characteristic. There results a rapid method for determining the order of certain integral functions and a general result is suggested. From this result Fourier's formula for all the analytic solutions of the equation considered can be established. Analogous results are obtained for other equations.

VOLTERRA (Rome): "Sopra il metodo delle imagini nelle equazioni del tipo iperbolico."

The author started from the formulas given by him in the *Acta Mathematica*, volume 18, for the integration of the fundamental equations of the hyperbolic type. He recalled that he had given the method of images in particular cases in his Stockholm lectures and finally showed that the principle of images can be considered in all its generality in the case of equations of the hyperbolic type, exhibiting the peculiar character which the method of successive images assumes, in virtue of the influence which the real characteristic cones exert.

ZERVOS (Athens): "Sur la correspondance entre la théorie d'intégration des équations aux dérivées partielles du premier ordre et l'intégration des systèmes de Monge."

The author showed how in certain particular cases the theory of the integration of partial differential equations can be made to correspond to the theory of the integration of a system of Monge.

MOORE (Chicago): "On a form of general analysis with applications to linear differential and integral equations."

Two principles of generalization were discussed, one by means of abstraction and the other by means of the addition of a parameter. These were enunciated and illustrated by means of the general theories, 1) of the linear integral equation, 2) of the linear differential equation. These general theories have the form of systems of deductive logic, and their special form suggests, as probably useful, the following scheme of general analysis. One branch of general analysis (of the form in question) is a system of deductive logic which is referred to the system of real numbers and comprises one or more variables, in whose character and field of validity nothing is fixed except by means of postulates regarding one or more symbols not defined, which represent classes of functions of those variables.

FREDHOLM (Stockholm): "Les intégrales de Fourier et la théorie des équations intégrales linéaires."

The author calls attention to the fact that a class of integral equations of the second kind exists whose solution is a polyform function from which are derived all the integral equations of the first kind which are solved in the manner of the integrals of Fourier.

D'Adhémar (Lille): "Sur les équations intégrales de Volterra."

The author showed the necessity for introducing the partifinite of the infinite integrals in the problem of Volterra. He treated a problem a little more general than that of Volterra and Lalesco by the method of successive approximations and studied the analytic nature of the solutions.

Orlando (Rome): "Sulla risoluzione delle equazioni integrali."

The author showed a method of solving integral equations with a polynomial nucleus and by means of the method of successive approximations passed to the general case. He also considered equations which contain the derivative of the unknown function.

Pascal (Naples): "Sulla nuova teoria delle forme differenziali di ordine e grado qualunque."

This gave a summary of all the principal results which the author had obtained for this theory up to the present time.

De Donder (Brussels): "Sur les invariants intégraux." (Presented by title.)

Stéphanos (Athens): "Sur une extension de la théorie des covariants et invariants de formes binaires."

The author considered the expression $\sigma(y, y', y'' \cdots)$, where y denotes any function whatever and y', y'' its successive derivatives, having the property of satisfying all differential equations

$$my\frac{\partial\sigma}{\partial y'} + (m-1)y\frac{\partial\sigma}{\partial y''} + \cdots + (m-\rho+1)y\frac{\partial\sigma}{\partial y^{(\rho)}} + \cdots = 0.$$

These expressions are generalizations of the covariants of binary forms. The author gave different properties of such expressions and showed their importance in the solution of different algebraic problems.

Montessus (Lille): "Sur les relations de recurrence à trois termes."

The author refers to the solution of the problem of analytic continuation which the theory of continuous functions gives, and seeks to demonstrate Poincaré's theorem by substituting a relation of recurrence with constant coefficients for the relation of recurrence with variable coefficients.

PUCCIANO (Demetrio Corace): "Contributo alla critica di alcune questioni che si riattaccano all'equazione differenziale di Laplace."

The author demonstrated: 1) That the conditions sufficient for asserting the integrability of the partial differential equation of the first order $\partial v/\partial x + \partial v/\partial y = f(x, y)$ are also sufficient for the integration of the differential equation of Poisson $\Delta_2 u = h(x, y)$ provided that v is capable of definite integration in a domain σ about the point (x, y) and provided that in the same domain $\partial^2 v/\partial x \partial y$ and $\partial^2 v/\partial y \partial x$ exist, are equal, and capable of integration. 2) If the function g_s of the arc of a closed line is such that the logarithmic potential $\int g_s \log 1/r \, ds$ be transformable, in the region outside, into a logarithmic potential $\int n_s \frac{\partial \log 1/r}{\partial u} ds$, there exists, in the region inside, the harmonic function which has on the boundary the normal derivative equal to g_s.

CAPELLI (Naples): "Sopra i coefficienti degli sviluppi delle funzioni algebriche."

NICOLETTI (Pisa): "Riduzione a forma canonica di un fascio di forme bilineari e quadratiche."

The author gave the fundamental lines of a rational demonstration of the theorem of Weierstrass on the equivalence of two pencils of bilinear forms, whose determinant is not identically zero, considering particularly the case of pencils of symmetric forms.

FUBINI (Genoa): "Sulla teoria dei gruppi discontinui."

After a summary of the results of Poincaré, the author gave some generalizations of groups which transform any region whatever into itself, of groups of conformal transformations, and finally sketched a method which in some cases may serve to recognize the discontinuity of linear groups in any number of variables.

DICKSON (Chicago): "On the last theorem of Fermat" (presented by title).

LEVI, B. (Cagliari): "Sopra la equazione indeterminata del 3° grado."

The author occupied himself with the determination of the rational solutions of the indeterminate equation of the third degree by means of rational processes, starting from known rational solutions: particularly with the case in which such processes reproduce the solutions with which one started. He defines the principal cases and finally connects the problem with that of the solution, in numbers rational with respect to the modulus of the equation, of the division of the period of the elliptic sine.

FRATLINI (Rome): "La nozione di indice e l'analisi indeterminata dei polinomi interi."

After a brief introduction, the author separated the irrational binomials $E + F\sqrt{D}$ (where E, F are rational functions) into hyperbolic and elliptic, and defined the index of an irrational binomial, giving a rule for its determination. He concluded by giving some applications.

SEVERINI (Catania): "Sulle successioni infinite di funzioni analitiche."

Having recalled the researches made on the subject in the direction of Cauchy and Riemann, the author passed to the consideration of it from the point of view of the theory of Weierstrass, in the way, that is, of making the property of the given succession of analytic functions depend on the conditions given for the elements which determine them.

ZAREMBA (Cracovie): "Sur le principe de Dirichlet."

The author substitutes for the problem of Dirichlet another more general problem which he calls the transformed problem and shows that the theory of Dirichlet's principle and of allied questions result very simply from the theory of the transformed problem.

BOGGIO (Turin): "Sulla risoluzione di una classe di equazioni algebriche che si presentano nella matematica finanziaria."

Professor Boggio calls the attention of analysts to the solution of those algebraic equations in which all the coefficients are positive, except the known term which is negative. Such equations are met in the general theory of loans; and the determination of the return from a loan reduces itself exactly to finding the unique positive root of equations of this kind.

SECTION II. GEOMETRY.

The section was opened by Segre; the following acting as chairmen at different meetings: Zeuthen, Darboux, d'Ovidio, Schur.

The following papers were presented:

ANDRADE (Besançon): "Le théorème de Ampère-Stokes et le postulatum d'Euclide."

If one considers a non-euclidean translation along an infinitesimal closed curve which is the boundary of an element of area as equivalent to a normal rotation proportional to the area, there follows from this an intuitive demonstration of Archimedes's theorem on the equilibrium of pressure distributed uniformly on a closed surface. This demonstration suggests the immediate extension of the theorem of Ampère-Stokes to non-euclidean space.

One deduces from it that the non-euclidean is the only space which admits an orthogonal net of "dallage," and establishes that the vectorial element $ds^2 = dx^2 + dy^2 + dz^2$ is characteristic for euclidean space.

The result is not new but the connection with the theorem of Ampère-Stokes is interesting.

VARICAK (Agram): "Beitrag zur nicht-euklidischen analytischen Geometrie."

Starting from the translation of Halsted relative to the geometry of Bolyai-Lobachevsky and from a work by Liebmann on the same subject, the author gave some applications of the calculus in the analytic geometry of Lobachevsky.

ZEUTHEN (Kopenhagen): "Un esempio di una correspondenza senza Wertigkeit."

The author spoke of the determination of pinch points of the ruled surface generated by the lines which cut a given line.

MONTESANO (Naples): "Sui complessi bilineari di coniche nello spazio."

The author gave an account of some researches begun in

1888, and made mention of the studies of Humbert; he gave the common genesis of the complexes which he has studied.

Severi (Padua): "Di alcuni recenti risultati sulla geometria algebrica e di qualche problema ad esso collegato."

Having recalled the known results relative to the simple integrals belonging to an algebraic surface, the author considers the cause of irregularity of a linear system which is connected with the irregularity of the surface, for the purpose of finding a neat determination of the cause of the irregularity noted. He defines on an algebraic curve of the surface the reduced canonical series and gives the geometric genesis by demonstrating that the independent curves $C + C'$ which pass through the base points and through the double movable points of the curve of a pencil $|C|$ are $p_g - p_a$ in number, $|C'|$ being the system adjoint to $|C|$ and $p_g - p_a$ being the irregularity of the surface. These curves $C + C'$ cut a curve C, outside of the base points, in the reduced canonical series.

The preceding result is obtained as a consequence of the following theorem of which the demonstration was sketched:

The necessary and sufficient condition that an abelian integral of the second (or of the first) kind belonging to a curve whose coefficients contain rational parameters have periods independent of the parameters is that the integral considered should not become of the third kind for any particular value of these parameters.

Bagnera (Messina): "Sopra le equazioni algebriche $f(x, y, z) = 0$ che si possono risolvere con x, y, z, funzioni meromorfe quadruplamente periodiche di due parametri."

Given the table of primitive periods

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ a_1' & a_2' & a_3' & a_4' \end{vmatrix}$$

for the quadruply periodic functions of two parameters u, v

$$x = p_1(u, v), \quad y = p_2(u, v), \quad z = p_3(u, v),$$

and supposing that to every point (x, y, z) of the surface corresponds a pair of values (u, v), one can consider the surface as representing the groups of an involution on an hyperelliptic surface of the given table. The linear substitutions.

$u' = au + bv + c$, $v' = a'u + b'v + c'$ which transform a point (u, v) into a point (u', v') belonging to the same group with (u, v) form a group.

De Franchis (Parma): "Intorno alle superficie regolari di genere uno che ammettono una rappresentazione parametrica mediante funzioni iperellittiche di due argomenti."

This was an exposition of the methods which serve for the effective construction of the regular hyperelliptic surfaces of genus t. The criteria were also given which serve to characterize these surfaces.

Radus (Budapest): "Über die Wendetangentenebenen von Raumkurven."

The author gave some criteria for the existence of inflexional tangent planes for twisted curves in ordinary space and space of higher dimensions.

Bianchi (Pisa): "Sulle trasformazioni di Darboux della superficie di area minima."

The transformations of minimal surfaces are those given by Darboux in volume 3 of his Théorie des Surfaces (page 441). From a given surface S of minimum area, by the transformation D of Darboux, we obtain a simple infinity of such surfaces. These transformations applied to real surfaces give imaginary surfaces.

These transformations of Darboux preserve the lines of curvature and the minimum lines. When they are compared with those (real) transformations of minimal surfaces which result from the theorem of Guichard on the deformation of a paraboloid of revolution, we observe that every such transformation of Guichard can be resolved into two conjugate imaginary transformations of Darboux.

Four classes of applicable surfaces depending on minimal surfaces were considered, viz.:

(*a*) The evolutes of the minimal surfaces.

(*b*) The deformed surfaces of the paraboloid of revolution.

(*c*, *d*) The surfaces applicable to the two paraboloids of Darboux with a generator tangent to the circle at infinity.

For the surfaces of each class the transformations of Darboux give rise to a line congruence W whose two focal surfaces belong to the class considered.

The transformations thus obtained for the deformed surfaces of the paraboloids are only a particular case of the transformations for the surfaces applicable on the general quadric, of which a hint is given by enunciating a more general problem whose solution seems desirable.

PANELLI (Rome): "Sopra un carattere delle varietà algebriche a tre dimensioni."

Given in a variety V_3 a net $|S|$ of surfaces S determined by a simple base curve of genus π and by a number P_0 of base points, also simple for each surface S. Call P the arithmetic genus of the (variable) curve of intersection of two surfaces and P_0 the arithmetic genus of a surface S. The expression

$$I_1 = g + P_0 - gP_1 - 36P_0 + (\pi - 1) - 28,$$

in which g denotes the genus of the Jacobian of the given net, does not depend upon the selection of the net and for this reason constitutes a relative invariant of the variety.

The invariant I_1 and Segre's invariant I_0 of the same variety V_3 are connected with the arithmetic genus π_θ by the relation

$$48\pi_\theta - 56 = 2I_1 - I_0.$$

Between the invariants I_0, I_1 and Ω the following relation exists:

$$2L(\Omega_0 - \Omega_1 + \Omega_2) = 2I_1 - I_0 - 42.$$

The invariant I_1 is applied to the solution of problems in enumerative geometry concerning the nets Ω of surfaces contained in a V_3.

DINGELDEY (Darmstadt): "Zur Erzeugung der Kegelschnitte nach Braikenridge und Maclaurin."

The author showed how to classify the elements which serve for the generation of the conic sections according to Braikenridge and Maclaurin, in order to generate the hyperbola and the parabola in the simplest manner possible. Also the dual generation, according to which these curves appear as the envelope of their tangents, was recalled.

FINSTERBUSCH (Zwickau): "Über eine Erweiterung eines Schliessungsproblems von J. Steiner und ihre Beziehung zur Gauss'schen Theorie zentrierter Linsensysteme."

Two extensions of the problem of Steiner were given: The first extends the theorem of Steiner to the cyclides of a pencil and reaches the final form

$$\frac{u_2}{Z_{\kappa_1}} + \frac{u_2}{Z_{\kappa_2}} = \tfrac{1}{2} - \frac{r}{Z},$$

which for $r = 0$, $Z = 1$ gives Steiner's formula. The second extension consists of a general theorem which presents an analogon to the theory of central optical systems.

GALLUCCI (Naples): "Su la configurazione armonica."

The 24 points and the 24 planes of the configuration can be indicated by means of the 24 permutations of 4 elements and the properties of the configuration result immediately.

BRÜCKNER (Bautzen): "Bemerkung zur Morphologie der aussergewöhnlichen Polyeder erläutert durch die Sechsfläche."

Indicating on the blackboard the images of the polyhedrons with six double or single faces, the principles of the division of the irregular polyhedrons into classes and the principal methods for the generation of these forms were explained.

BROUWER (Amsterdam): "Une théorie des groupes finis et continus indépendante des axiomes de Lie."

In this paper the theory of Lie's groups is founded on the existence of certain derivatives of functions determining the group, then the existence of the infinitesimal transformations is deduced from this. The biuniform and continuous representation of the groups on a continuum of P dimensions was shown and from this the groups of the varieties of one and two dimensions were determined.

P 35,780

TZITZEIKA (Bucarest): "Sur une nouvelle classe de surfaces."

Starting from a system of partial differential equations which the coordinates of a point of a surface referred to general curvilinear coordinates satisfy, it is found that the surfaces for which the ratio of the total curvature to the fourth power of the distance from the origin to the tangent plane remains constant, are the simplest surfaces which present themselves in this theory.

PFEIFFER (Kiev): "Des développements des fonctions algébriques de deux variables indépendantes en séries entières des variables indépendantes."

The condition was found for the existence of algebraic functions of two independent variables in series of powers of the independent variables in uniplanar and biplanar points.

SECTION III A. MECHANICS AND MATHEMATICAL PHYSICS.

The section was opened by Professor Pizzetti; the presidents of the various sessions were Professors Volterra, Darwin, Liapounoff, Wangerin, Luiggi. The programme contained the following papers:

DARWIN (Cambridge): "The rigidity of the earth."

According to the theory of Lord Kelvin, if the earth were as rigid as steel the oceanic tides would have an amplitude two thirds of that which they would have on a globe absolutely rigid. The author examined the grounds of this conclusion, compared them with the older theory of Laplace, and made various corrections by means of the investigations of Rayleigh, Paschwitz, Ehlert, Cortazzi and Hecker. The conclusion is that the ratio is about .6, that the coefficient of elasticity is different in different directions, and that the solid earth rises and falls about 17 cm. by the action of the tides.

LAMB (Manchester): "The flexure of narrow beams."

LAURICELLA (Catania): "Sull'equazioni $\Delta^{2n} V = 0$ e su alcune estensioni dell'equazioni dell' elasticità."

The method indicated by Professor Lauricella to demonstrate the possibility of solving the problem of the boundary values for the equation $\Delta^{2n} V = 0$ in any number of variables is valid also for the equation $D^{2n} V = 0$ and in particular for the equation $D^6 V = 0$.

For this purpose a theorem of equivalence is demonstrated under which the problem is reduced to that of finding the solutions of a system of equations in more unknown functions which take given values at the boundary, a system in fact analogous to that of the equations of elasticity. For this system of equations he introduces the concept of pseudo-hypertensions, which are integrals analogous to double distribution, and which lead to the consideration of a system of functional equations to which one can apply the known results of Fredholm. In this manner he demonstrates the existence theorem for the problem stated above.

SOMIGLIANA (Turin): "Sulle deformazioni elastiche non regolari."

The author demonstrated first that elastic distorsions can exist satisfying the conditions established by Weingarten and not satisfying those of Volterra, and that such distorsions are possible also in bodies simply connected. Making use of a concept of Professor Morera, he announces a possible theory of the state of tension of "Prince Rupert's drops."

ABRAHAM (Göttingen): "Zur Theorie der Wirbelstrombremsen."

On the basis of the Electrodynamics of Hertz for bodies in motion, the field on the inside of a moving body was determined when that body was composed of iron or copper, which acts as a brake. The author investigated how far the dependence of the force on the velocity can be determined theoretically.

ANDRADE (Besançon): "Sur une nouvelle méthode de mesure des frottements."

The measure of the coefficient of friction is made by the author by means of the study of the extension of the movement of an oscillating wheel in which the friction comes from the action applied on the two cylindrical supports of different radii. The resistance of the air is eliminated and thus also is eliminated the friction of rotation, the measurements being made on supports of different radii.

KORN (Vienna): "Über die universellen Schwingungen der Materie mit Anwendungen auf die Theorie der Gravitation und der intramolekularen Kräfte."

After a brief exposition of the fundamental principles of universal vibrations — vibrations of the material particle slightly compressible, immersed in a half compressible fluid — the vibration of pulsation necessary for the explanation of gravitation as the fundamental vibration of the universal vibration was explained; and for electrons which must be considered also as pulsating particles a mechanical difference from gravitating particles was found, this difference being expressed by substituting Coulomb's law in place of Newton's.

LEVI-CIVITA (Padua): "Sull' espressioni assintotiche dei potenziali ritardati."

A retarded potential v, when the point P which caused it is moved, depends on the velocity of the movement. It is advantageous to consider the position of the body at an earlier time, but there is no advantage in doing so when the velocity of propagation increases without limit and when the trajectory of P passes through its position at the actual time considered. The paper is concerned with the development of the retarded potential in the second limiting case. The concept of Abraham of the quasi-stationary condition finds its mathematical expression in this asymptotic development.

GARBASSO (Genoa): "La luce bianca."

The author studied the normal spectrum of a damped vibration. In accord with Zelinder's experiments and in opposition to the theory of Carvallo, he succeeded in establishing that the spectrum considered is continuous. The method consisted in decomposing the function $e^{-at} \sin bt$ into a sum of undamped harmonics, but limited to a finite interval of time.

GREENHILL (London): "Geometry of the motion of the spinning top."

This was a method for designing a family of curves, hodographs of the Poinsot herpolhode, employing the focal ellipse of a deformable hyperboloid and a tangent generating line for the determination of the apsidal angle and of the limiting inclinations of the axis.

SOMMERFELD (Munich): "Beitrag zur Erklärung der turbulenten Flüssigkeitsbewegung."

The gap which seems to exist between practical hydraulics and theoretical hydrodynamics has been repeatedly observed and deplored. It is known that Osborne Reynolds succeeded in crossing this abyss by introducing the notion of stability. He brought to light the existence of a critical point for the velocity, rendering possible the qualitative law with dimensional considerations, and determined the numerical value by experimental means. Above the critical point the movement is unstable and below it is stable.

These ideas have been notably simplified and extended by the work of Lorentz, who also recognized that the case of the flow between two concentric cylinders and between two plane walls, studied experimentally by Conetta is still simpler than the typical problem of the flow in a cylindrical tube, originally taken for consideration. The research of Professor Sommerfeld relates precisely to this simpler case of Lorentz-Conetta and treats it by the classic method of small oscillations.

The general solution of the problem can be given without difficulty; the expression is found by means of the integrals of cylindrical functions. The condition of instability is characterized by a certain transcendental equation. The discussion of the equation, in the simple case of small velocity, leads to certain results already noticed by Lord Rayleigh.

BOGGIO (Turin): "Sopra alcuni teoremi di fisica matematica."

The author gave a direct method for demonstrating that the integrals of the indefinite equations of numerous problems of mathematical physics, when considered as functions of the parameters which figure in the equations themselves, cannot depend on such parameters, but rather on the simple poles.

BOCCARDI (Turin): "Sopra una nuova equazione nelle osservazioni dei passaggi."

Professor Boccardi has made a long series of observations of right ascension and has established that, reduced to the medial equinox, they present oscillations according to the state of the sky, in the sense that the right ascensions for observations with no wind and with sky entirely serene are smaller than those observed in bad conditions. He constructed a table of comparisons, which he called equations of transparency.

ANDRADE (Besançon): "La synchronisation par le fer doux."

The periodic synchronizing force does not exist with the use of soft iron, yet by Picard's method of approximation the theory can be extended to this case and also to that in which the viscous resistance is no longer exactly proportional to the velocity.

GENESE (Aberysthwyth): "The method of reciprocal polars applied to forces in space."

The note is an extension to space of a paper on reciprocation in statics, published in the *Proceedings of the London Mathematical Society* in 1886. As an illustration of the first note, the author demonstrates how a weight can be distributed on a closed curve in such a way as to put the centroid in any point whatever in the interior. He obtains the important result

$$\int \frac{ds}{p^2 \rho} x = 0$$

on each closed curve. He further showed how the fundamental formula of Grassmann is applied to the case in which e/c is not equal to unity; coordinates not being necessary.

MACFARLANE (Chatham): "On the square of Hamilton's delta." (Presented by title.)

TEDONE (Genoa): "Sopra il problema di Lamé."

Professor Tedone sought to give a complete solution of the problem of determining the triply orthogonal systems formed by the surfaces $q_1 = \text{const.}$, $q_2 = \text{const.}$, $q_3 = \text{const.}$, such that two of these series of surfaces be surfaces of rotation about the same line in such a way that Laplace's equation have the simple solution of the form

$$Pf_1(q_1)f_2(q_2)f_3(q_3),$$

P being a given function.

BRYAN (Upper Bangor, Wales): "Notes on the steering ot automobiles and on the balancing of ships."

The connections commonly used in the steering of automobiles do not completely satisfy the geometric conditions to prevent lateral slipping. The problem here considered tends to this end. Considering the two front wheels, the locus of their gyrating center is a curve of order eight and the condition demanded is that the axis of the rear wheels be the asymptote of this curve.

Chemien has recently discussed the possibility of eliminating the rolling of a ship by means of a pendulum or a movable mass contained in a case filled with a viscous liquid. It is here shown that such a method gives rise to a system with two degrees of freedom. The oscillations are determined by an equation of degree 4, and one must separate this equation into its quadratic factors in order to study the two oscillations. For this purpose one adopts the real root of the reducing cubic and approximates to this value for the case in which the mass of the pendulum is small compared with that of the ship. It is true that this arrangement becomes dangerous under certain circumstances.

POYNTING (Birmingham): "The momentum of a beam of light."

Considering the energy of the incident and reflected waves when a perfectly reflecting surface is moved toward a plane pencil of waves, the author deduces the ordinary formula in which the pressure of radiation of the plane waves is equal to the energy of volume, with the limitation that the waves are of such a character that the energy of volume is inversely proportional to the square of the length of the wave. Supposing that the pressure of radiation is due to the momentum of the pencil, the author obtains a different expression for the pressure on a movable plane. The last concept suggests the existence of a tangential component when the pencil falls obliquely on an absorbing surface. The existence of such a tangential component is confirmed by the experiments of the author.

KOLOSSOFF (Iuriev): "Sur le problème plan dans la théorie d'élasticité."

The problem leads to the integration of the differential equations

$$\frac{\partial N_1}{\partial x}+\frac{\partial T}{\partial y}=0, \quad \frac{\partial N_2}{\partial y}+\frac{\partial T}{\partial x}=0, \quad \Delta_2(N_1+N_2)=0,$$

in which

$$2T=\alpha\frac{\partial(N_1+N_2)}{\partial y}-\beta\frac{\partial(N_1+N_2)}{\partial x}+\phi,$$

$$N_1-N_2=\beta\frac{\partial(N_1+N_2)}{\partial y}+\alpha\frac{\partial(N_1+N_2)}{\partial x}+\psi,$$

ϕ and ψ being two conjugate functions and α and β satisfying the equations

$$\frac{\partial\alpha}{\partial x}-\frac{\partial\beta}{\partial y}=-1, \quad \frac{\partial\alpha}{\partial y}+\frac{\partial\beta}{\partial x}=0.$$

Various cases can be treated in a similar manner and the method indicated gives the mode of transforming the solutions obtained and of finding some new ones.

SECTION III*B*. ACTUARIAL SCIENCE AND ENGINEERING.

The section was opened by Professor Toja. Presidents: Toja, Quiquet.

TOJA (Florence): "Alcune considerazioni sui rapporti tra le matematiche e la scienza attuariale."

The author considered the problems of actuarial science connected in greater part with mathematics, dwelling more especially on the concept and determination of probability a posteriori, from the perequation and from the analytical representation of an observed phenomenon. He then discussed the solution of three complex problems of the mathematical theory of pensions which furnish a good idea of actuarial devices.

QUIQUET (Paris): "Sur une nouvelle application des jacobiens aux probabilités viagères."

The author explained that this paper is the result of his actuarial thesis (1892) and of the communication which he presented at the congress in New York (1903).

POUSSIN (Paris): "Sur l'application du graphicisme aux calculs d'assurance."

The paper of Poussin is intended to complete a paper published in 1904, in the *Bulletin* of the French actuaries. The case discussed is that of an endowment insurance with counter insurances. By opportune transformations the author succeeds in putting the analytic expression of the premium in such a form that it can be calculated graphically with appropriate wire polygons. An album containing the necessary tables for performing the various graphical processes was exhibited.

ELDERTON (London): "A comparison of some curves used for graduating chance distribution."

In this paper the principal systems of curves of frequency studied by Pearson, Thiele, Charlier, Edgeworth, Bruns were exhibited. After having outlined the method of perequations of Pearson, the author applied equations of curves of form already known to practical cases relative to typical frequency.

BOHLMANN (Wilmersdorf-Berlin): "Über die Grundlagen der Wahrscheinlichkeitsrechnung in ihrer Anwendung auf die Lebensversicherung."

Mr. Bohlmann forms a system of postulates from which theorems of the calculus of probability can be deduced logically. He examines the criteria of probability concerning the independence of events and demonstrates the sufficiency of the usual criterium. He formulates supplementary postulates necessary to define the survival functions of the mathematics of insurance and gives a relation between the survival functions of groups and those of single individuals.

Borel (Paris): "Sur les applications du calcul des probabilités aux sciences biologiques."

It is desirable to make the same distinction in the science of biology as in mathematical physics between the two following questions:

(*a*) The investigation of the rigorous relations between abstract values.

(*b*) The study of the approximate relations between these abstract values and the concrete values furnished by experiments.

March (Paris): "Une nouvelle statistique internationale de la population. Observation sur la comparaison et sur la terminologie des statistiques." (Read by Borel.)

De Helguero (Rome): "Sulla rappresentazione analitica di alcune statistiche."

Observing that the methods proposed for the study of arithmetical statistics do not harmonize with the normal curve, the author gives some methods for a study in which they are considered as perturbations of the normal variables.

The principal hypothesis is that in which the equation

$$y = \frac{y_0}{-\sigma\sqrt{2\pi}}\left(1 + \frac{a}{\sigma}(x - b)\right)e^{-\frac{1}{2}\left(\frac{x-b}{\sigma}\right)^2}$$

is assumed; the author has constructed some tables for the calculation of the parameters.

Lembourg (Brussels): "L'actuaire, sa fonction, les deux aspects de celle-ci."

The actuary, like the engineer, has need of mathematics: but the practice of his profession can render some contributions to mathematics itself. In fact, the actuary does not limit his function to the study of questions of insurance, but all fields which can be explored only by the aid of statistics are of interest to him.

The question presents two aspects — that previously given and this other: the mathematical spirit which he possesses causes him to introduce into the solutions of the financial questions a correctness and a rigor which result in good both for the insurer and the insured.

GINI (———): "La regolarità dei fenomeni rari."

The author spoke of the regularity of rare phenomena and proposed to demonstrate how the theoretic reason given by Bortkewitsch for a normal dispersion of the series of rare phenomena does not withstand criticism. He gave the results obtained from the measure of the regularity of some statistical series of rare phenomena relative to mortality.

DAWSON (New York): "Necessary cautions in dealing with actuarial problems."

In the absence of the author the communication was read by title and copies of the printed paper were distributed.

CASTELLI (Rome): "Sull'insegnamento della matematica attuariale e finanziaria nelle scuole professionali inferiori, medie, e superiori."

The paper more especially dealt with the work of the minister of agriculture, industry and commerce in Italy; it proposed to promote the study of the applications of mathematics to questions of credit, foresight, and economy in the professional schools.

SECTION IV. PHILOSOPHY, HISTORY AND TEACHING OF MATHEMATICS.

Section opened by Professor Enriques. Presidents: Enriques, Loria, Vailati, Fehr, Zeuthen, Picard, Simon.

At the opening Professor Enriques delivered an address: "Matematica e filosofia."

He showed the development of the philosophic ideas dominant among reflecting mathematicians in regard to the conception which they form of real science. In the first period (cartesian) the cosmologic problem is accompanied by a metaphysical concept; after the newtonian systematization the problem of knowledge dominates (Kant); in our days the critical spirit tending to a rigorous foundation of mathematics is the principal characteristic. These three principles correspond to a purification and abstraction of mathematical theory which combine, in origin, the elements of experience, intuition, and logic. In fact from a statical point of view the complete mathematical theory tends now to a formal orientation which satisfies certain esthetic and economic demands.

But side by side with this point of view is the dynamical view of the science of the investigators who regard the theory in formation. Psychological problems accompany this view. The duality between the logical and psychological tendencies of reflecting mathematicians today corresponds exactly to the different ideals which these thinkers form of science, as a complete organism or a formative process.

HESSENBERG (Bonn): "Numeri ed intuizioni."

The fundamental theories of numbers are essentially theories of type ω. This is an object of logic, but the construction of their elements results from intuition, since these elements are logically irreducible. The case is perfectly analogous to that of geometry in which the system of relations is the logical object but the elements themselves are not.

BOUTROUX (Paris): "Sur la relation de l'algèbre à l'analyse mathématique."

Historically mathematical analysis is an extension of algebra. Analysis and algebra were synonymous terms for Newton, Euler, and Lagrange, the former differing from the latter in so far as it implies infinite operations; thus a long tradition leads us to consider analysis as the study of convergent algebraic expressions. This opinion does not now seem to be tenable. Analysis is not a construction; it is the effort we make to analyze and to translate mathematical laws into the language of algebra. Algebra is only an instrument of analysis.

ITELSON (Berlin): "Logik und Mathematik."

We can determine the relation between logic and mathematics only when we are given a definition of one or the other. Logic is not, as ordinarily defined, the science of thought; it is, when we actually consider it, the science of things in general.

The author spoke also of the calculus of logic, of the duality of the operations, of the invariator, and of the absorptor.

ITELSON (Berlin): "Deduction, Induction, und Perduction."

The author spoke of the office of deduction and induction in mathematics. The inference from n to $n + 1$ is very improperly called complete induction; it consists of two reasonings, one complete induction and one deduction; on account of this it merits a special name. The speaker proposed the name perduction.

SIMON (Strassburg): "Du continu, point, et ligne droite, remarques historiques."

The speaker referred to Galileo and Leonardo as the ones who had first recognized exactly the problem of the continuum; he then proceeded to dispute the arithmetization of the continuum made by Cantor, pointing out that the fundamental series of Cantor defines an arithmetic segment. The speaker remarked that he agreed with Veronese. He then referred to the two axioms on continuity given by Hilbert, and declared that for geometric geometry they say nothing. He then spoke of points and lines, and concluded by declaring that he did not believe it possible to formulate with determinate propositions the mental labor which embraces periods of very long time.

BERNSTEIN (Göttingen): "Nachweis, dass unter allen Beweisen des Pythagoräischen Lehrsatzes der Beweis des An-Nairizi (900 n. Chr.) der axiomatisch einfachste ist."

If we count as a criterion of simplicity the number of applications of the congruence axiom in the plane we have the precise result that a demonstration does not exist, for addition, which makes use of less than six applications. From this it follows that in the sense fixed, the demonstration of An-Nairisi (900 A. D.) is the simplest. This is an example of the method by which we can define the concept of simplicity of a demonstration, an example however which is not unique.

PASTORE (Aosta): "La natura extralogica delle leggi di tautologia e di assorbimento nella logica matematica."

The principal feature of this paper is the demonstration of the extralogical nature that is purely descriptive of the laws indicated. From such facts results the possible introduction in mathematical logic of a group of new notions and operations, among which the most noteworthy is that of multiples and powers universally rejected from Leibniz until the present time.

LORIA (Genoa): "Le tradizioni matematiche dell'Italia."

This was the presidential address which Professor Loria delivered at the second session of the section. After greeting the members he gave the section a taste of the studies undertaken by him in order to continue the "Histoire des sciences mathématiques en Italie." Waiving the possibility that the history of Italian mathematics should begin with Pythagoras, Archimedes, and Archita, the author considers the beginning as 1200, when the "Liber abaci" of Leonardo Pisano appeared, to which he assigns much importance. He explained why Fibonacci had not had any immediate disciples by the fact that in the three succeeding centuries art absorbed the better minds; he touched lightly on the humanistic period and on the rising and flowering of mediaeval universities. He then said something of the golden period of Italian algebra, commenced with Luca Paciuolo and closed with Bombelli and Cataldi, noting how the discovery of the "punto di concorso" made by Guidobaldo del Monte opened a new era. The following century was that of Galileo Galilei, to whose school belong also Cavalieri, Torricelli, Borelli, and Viviani, to whom geometry owes so much. The author then passed to a description of the analytic works of Manfredi, Fagnani, and Ricatti, mentioning also G. Saccheri and G. Malfatti. In the century in which these flourished J. L. Lagrange was born. The speaker demonstrated that the latter should be counted as an Italian mathematician. He made mention also of Mozzi, Mascheroni, Ruffini, Fergola, Lorgna, and finally of the line of historians headed by Cossali. Passing to the nineteenth century, the speaker described the reception which descriptive geometry had in Italy, enumerating those who made it progress in Italy and was thus led to speak of the Italian geometric school founded by Luigi Cremona. That Italy was not a passive spectator of the development of analysis in the century preceding ours, is demonstrated by the schools which flourished at Turin, Pavia, and Pisa. In closing he made brief mention of the other illustrious mathematicians whose native land was Italy, noting that in no epoch were the Italian geometers isolated and that the history of Italian mathematics developed without interruption or lull; therefore there is hope that it will continue to flourish in the future.

ZEUTHEN (Kopenhagen): "Sur les rapports entre les principes anciens et modernes de la géométrie."

The author observed that each of the two groups of such principles is an inseparable logical system. In order to understand the ancient principles he considered their real point of departure. From this point of view he sought to show their logical value.

SMITH (New York): "The Ganita-Sara Sangraha of Mahâvirâcârya."

Professor Smith recently visited Japan, China, and India for the purpose of procuring material illustrating the history of mathematics. In this paper he referred to the Indian treatise of the ninth century, of which Professor Rangacharya of Madras is preparing a translation which will be ready in two years. Mahàviràcàrya lived at Mysore about 850. His work is principally algebraic, but comprises also notions of arithmetic and of measure. The most interesting features of this treatise have reference to certain parts of a series, to special forms of non-rational equations and to indeterminate equations.

The publication of this work will be interesting for the history of mathematics for the light which it will shed on the source of the works of Bhaskara, on the relation between the schools of Pataliputra and Ujjain and others of India, and on the influence of Greek mathematics on that of India and of this on that of Arabia.

Duhem (Bordeaux): "Sur la découverte de la loi de la chute des graves."

Professor Duhem called attention to the work of Albertus de Saxonia (De caelo et mundo) as that in which for the first time definite hypotheses were enunciated on the mode of variation of the velocity of falling bodies, and mentioned the possibility of the above work having influenced the ideas of Leonardo da Vinci.

Giacomelli (Rome): "I risultati di alcune ricerche sull'opera meccanica di Galileo."

The author showed the state of the historic questions on Newton's first two laws of motion, which, as is known, are due to Galileo. He showed what was the mental and experimental procedure used by Galileo to derive these laws. From the evidence it would result first that Galileo arrived at the proposition of the persistence of motion by making use of an idea already admitted by him a priori; in the second place that those experiments on the inclined plane by which he is believed to have been led to the discovery of the law of inertia, rather led him to the discovery of the independence of motions; finally that the relations between the two procedures were of purely external nature.

Pittarelli (Rome): "Luca Pacioli usurpò per se stesso qualche libro di Pièro de Franceschi?".

The author made a minute comparative study between the

Libellus Petri Pictoris Purgentis de quinque corporibus regularibus (copy existing at the Vatican) and Tractatus primus, secundus, tertius, which preceded the Divina proportione of Luca Pacioli and showed that these were indeed plagiarisms. He then gave a brief account of the Libellus from which it appears that Piero de Franceschi, artist, knew the geometry of Euclid quite well, and knew how to make use of algebra to solve geometric problems.

Gutzmer (Halle): "Über die Reformbestrebungen auf dem Gebiet des mathematischen Unterrichts in Deutschland."

The author gave a sketch of the work of the Unterrichts-Kommission der Gesellschaft Deutscher Naturforscher und Ärzte, indicating the general lines of the project of reform of this commission, which was established after the Congress of Heidelberg and which has lately finished its labors.

Borel (Paris): "Les mathématiques dans l'enseignement secondaire en France."

Professor Borel began by showing how the secondary instruction in France is divided into various sections as shown in the following scheme:

First Cycle. 11–14 years.	Second Cycle. 15–17 years.
	A Latin, Greek,
A	*B* Latin, Modern Languages,
	C Latin, Sciences.
B	*D* Science, Modern Languages.

In sections *A*, *B* of the second cycle the instruction in mathematics is extremely reduced and for this reason the speaker dwelt only on the sections *C* and *D*. He showed how some concepts had been introduced in these sections which before had been reserved for a later period, viz., the concept of the derivative as the slope of the tangent, and the notion of movement, concepts which Professor Borel holds are not more difficult than the comprehension of the Pythagorean theorem. He spoke also of the definition of cosine by means of the right triangle.

Godfrey (Winchester): "The teaching of mathematics in the English public schools for boys."

The author being absent, the paper was given in outline by President Vailati.

SMITH (New York): "The teaching of secondary mathematics in the United States."

The author explained that in this communication he had in mind the following purposes: 1) To put briefly in relief the historical influences which have contributed to the present state of mathematical studies in America; 2) To make known the present conditions of mathematical instruction in the United States; 3) To call attention to the influence which is now operating to give new form to the secondary instruction; 4) To consider some of the new directions; 5) To suggest some questions which a section, as the present, might take into consideration with advantage in an international congress by means of the formation of a committee which should represent the principal nations interested.

This last point should be taken into consideration by those who shall have the care of preparing the program of the next congress, since the probable stimulating influence of an international committee on the betterment of mathematical instruction will be such as to justify the attempt.

SUPPANTSCHITSCH (Vienna): "L'application des idées modernes à l'enseignement secondaire des mathématiques en Autriche."

It is necessary, according to the idea of Professor Suppantschitsch, to introduce into the secondary instruction the notions of function and derivative, establishing these in the first place without the notion of limit. In Austria experiments are actually in progress which will lead, without doubt, to a definitive rearrangement.

BEKE (Budapest): "Über den mathematischen Unterricht in Ungarn."

The Hungarian reforms are connected with those proposed by the German commission mentioned by Professor Gutzmer. A commission has also been appointed in Hungary which is occupied with the questions regarding instruction and which contemplates a reform in the instruction in mathematics by means of simplification of the programs and introduction of graphical exercises, by means also of the introduction of the concept of function and with a more intimate contact with practical applications, by means of the fusion of stereometry with descriptive geometry, and finally by means of the development of the sense of economy and with the introduction of the fundamental concepts of the differential and integral calculus.

VAILATI (Rome): "Su alcuni caratteri degli attuali programmi per l'insegnamento della matematica nelle scuole secondarie."

Professor Vailati observed above all that, according to the present programs, the exercises on the solution of equations come too late in the instruction in algebra and that instruction in geometry is not put in intimate relation with the instruction in drawing. He also spoke of the utility of introducing the concept of the derivative into the instruction in the higher secondary schools.

MARCOLONGO (Naples): "Sopra un trattato inedito di meccanica di Vincenzo De Filippis anteriore alla Mécanique Analytique di Lagrange."

The author spoke of a mathematician of Calabria, minister of the "Republica Partenopea" in 1799, who died a victim of the Bourbon reaction at the end of that year.

De Filippis had prepared a treatise on mechanics which was certainly written before the Mécanique of Lagrange and which the author has been able to examine among the manuscripts which are still preserved by the heirs. He mentioned several noteworthy things about this treatise and signalized above all the fact that De Filippis had attempted a general demonstration, for rigid systems, of the principle of virtual work.

STÉPHANOS (Athens): "Les mathématiques dans l'enseignement secondaire en Grèce."

After having traced in a few words the state of instruction from the capture of Constantinople to the epoch of the liberation of Greece, Professor Stéphanos spoke of the programs of mathematical instruction in Greece, and of the reform introduced in 1907 through the discussion and passage of a new law on text books.

ARCHENHOLD (Berlin): Über die Bedeutung des mathematischen Unterrichts im Freien in Verbindung mit Reformvorschlägen für den Lehrgang."

The mathematical conceptions should be deduced from observations of reality; this is exactly the principle of instruction. The different kind of angles should be presented by indicating the trees, considering houses, street crossings, etc. No problem should be solved without giving rigorous proof, but the concepts which are applied in the demonstrations must be acquired in principle by the use of real objects, not models. The simplest apparatus for measurement should find application at once to rectify the judgments, often fallacious, of the eye.

With such a method one can give in ten hours what otherwise would take years.

ANDRADE (Besançon): "Quelques observations psychologiques recueillies dans les enseignements scientifiques d'initiation."

Taking as example four very particular observed facts, Professor Andrade invoked a reform in the teaching of geometry, which he would reduce to two books: in the first the triangles, the dihedral, the trihedral; in the second, similitude and measure of extension. He wishes thus to limit the study of geometry because of the very narrow field of secondary instruction, which general culture does not permit to subordinate to preparation in special mathematics, and recalls on this point the profound truth expressed by Marcello Prevost in the "Lettere a Francesca" which should be read and meditated upon by the students of the normal school.

CONTI (Bologna): "Sulla iniziazione alle matematiche e sulla preparazióne dei maestri elementari in Italia."

For the lowest grades, the Italian educational legislation is still quite embryonic, without special instructions defining the limits, the programs, and the methods. If we desire that the primary grade does not prejudice the subsequent work of the school, it is necessary that that grade itself be disciplined in a manner that will serve for rational preparation for education and instruction.

The reforms tending to introduce in secondary instruction some mathematical concepts until now deferred, must be accompanied by others for these primary grades and for the elementary schools, in such a manner that the totality of the things taught in this first stage be a rational beginning for the subsequent studies.

One commits an exaggeration and falls into an equivocation when he opposes the statement that some special attitude for the assimilation of the most elementary concepts of mathematics is present in all pupils.

A rational beginning must be inspired by the general principles, defended by Froebel, Pestalozzi, Herbart, Chalotais, Vittorino da Feltre, and particularly by the mathematicians, by the principles on the basis of which Laisant recently republished his golden "Initiation mathématique." To such principles must the teachers of all grades of elementary schools be devoted. The children are now oppressed by a true and proper study; they are now required to do an annoying amount of memorizing, and are fettered by a continual formalism.

The actual arrangement of the normal school, the brevity of the course, the insufficiency of time assigned to mathematics, the defects of the present programs in rigor, and the imperfect functioning of the novitiate, all show that the mathematical preparation of elementary teachers is not adequate to the delicacy and importance of the mission which is confided to them. Therefore it is to be hoped that the rearrangement of the normal schools, imposed by the law of July 8, 1904, will not longer be delayed.

FEHR (Geneva): "Les mathématiques dans l'enseignement secondaire en Suisse."

After giving a summary of the organization of the higher secondary instruction in Switzerland, Professor Fehr showed the place which mathematics occupies, principally in the classic and technical sections. It is to be observed that in the classical gymnasia four hours per week is given to mathematics, in all classes, and that there for a long time trigonometry, plane and spherical, and analytic geometry have had a place, and consequently also notions of simple functions. In the technical sections of the gymnasia all branches of mathematics treated in the classical sections are studied more thoroughly, and also descriptive geometry and the elements of the differential and integral calculus are taught.

GARCIA DE GALDEANO ZOEL (Saragoza): "Alcune notizie sull'insegnamento matematico in Spagna."

The political struggle in Spain has retarded the progress of science and impeded the application of a new arrangement of public instruction. Nevertheless the founding of the Spanish scientific society, whose first act will be the mathematical congress of Saragoza next September, will be the beginning of a new scientific epoch. The speaker lamented that the elementary part of Spanish university instruction causes the higher instruction to be lost. He closed by making mention of the pedagogical means which, in his opinion, are the best for establishing an equilibrium between the progress of science and the intellectual potentiality for the study of science itself; these means he has defended in his publication, Alcunas consideraciones sobre filosofiá y enseñanza de la matematica, of which many copies were kindly offered to the congress.

GALLUCCI (Naples): "La questione logica e gnoscologica nei fondamenti della matematica."

The critique of the principles of mathematics belongs to mathematics and philosophy: whence a double point of view in their treatment; the logical point of view and the gnoscological point of view. The author gave a summary of the consequences of such a twofold view and mentioned as a possible conclusion that it might lead to a reelaboration of the theory of knowledge in which account is taken of the latest developments of the principles of science.

BROGGI (Rome): "Sui fondamenti del calcolo delle probabilità."

The author proposed a new descriptive definition (of the third kind) of the concept of probability and demonstrated the compatibility and the irreducibility of the system of postulates assumed to define probability.

EMCH (Solothurn): "Der Rechenkünstler Winkler und seine Methoden."

One of the most interesting personalities in the field of the art of calculating is Johann Jacob Winkler. While in various memoirs much is said of other calculators one finds almost nothing of Winkler; it is the purpose of this paper to tell something of this marvelous man, who was born February 23, 1831, in Wernatswill and died August 25, 1893, at the canton hospital of Stanz, poor and abandoned.

Winkler easily solved mentally every problem which presented itself in calculable form and utilized also all the rules of rapid calculation which can be deduced from algebraic formulas.

LORIA (Genoa): "Sur les moyens pour faciliter et diriger les études sur l'histoire des mathématiques."

Professor Loria, taking the cue from a recent article by Professor Eneström, noted that there are some persons who would be disposed to cultivate the history of mathematics if they should find some one who would guide them over the not easy road; and he noted how, the beginning of courses being difficult "ad hoc," the best system which presents itself is the compilation of a manual for researches in the history of mathematics, collective works whose plan would be determined conveniently enough by an international congress. In order to prepare the way for discussion, the author indicated the various subjects which, according to him, should find place in the work planned. He closed by presenting to the section one of his plans of an Indice delle materie.

AMODEO (Naples): "Appunti su Biagio Pelicani da Parma."

The author quoted from M. Cantor on the importance of knowing the content of the only known treatise published by Biagio da Parma, except for a very rare collection of mathematical treatises published at Vienna in 1515; and said that having had this collection in hand he has found in it two treatises of Biagio da Parma, and another treatise of Giovanni de Casali hitherto unknown.

From the reading of these two treatises of Biagio da Parma, he has found that his name should be written Pelicani instead of Pelacani, and that in Bologna this man had a dispute on the impact of rigid bodies which forms the subject of the treatise now found.

He gave an outline of the contents of the treatise entitled Tractatus de latitudinibus formarum Blasii de Parma, and concluded that in it the conception of Nicole Oresme on the latitude of the plane geometric form was enlarged and complicated by Biagio Pelicani by deviating from the straight road which would have enabled him to arrive sooner at the conception of the analytical geometry of Descartes, and that in this he was soon followed by his contemporaries, contributing to the putting aside of the simplicity of the question as it had been conceived by Oresme.

PITTARELLI (Rome): "Due lettere inedite di Lagrange all' Abate di Caluso esistenti nell' archivio storico municipale di Asti."

Professor Pittarelli found the letters in the Archive of Asti and now presents them (photographed for the occasion and given to the congress by the municipality of Asti). He was happy to give them into the hands of the two presidents, Professor Picard and Professor Simon, one from France and the other from Germany, nations which Lagrange, known and having become celebrated at Turin, rendered illustrious with his name in return for the hospitality which he had there. He asked that of the two copies one remain at the R. Accademia dei Lincei and the other with the faculty of science of the University of Rome.

AMODEO (Naples): "Sulla necessità di formare un archivio delle scienze matematiche."

In view of the difficulty of determining the author of a new proposition the historians of the science are often misled into assigning incorrect origins. The author emphasized the necessity of removing this difficulty, and urged the formation of a general scheme of archives of mathematical sciences. A detailed plan was submitted for adoption, but the section concluded that the congress should not yet undertake the enormous labor; it affirmed however the convenience of establishing such archives.

DE AMICIS (Brescia): "L'equivalenza (in planimetria) indipendentemente dalle proporzioni e dal circolo."

Considering substantially only the equivalence of parallelograms of equal bases and altitudes, the author established the theorem: If a triangle has its angles equal to those of a second triangle, the rectangle formed by a side of one and a side of the other is equivalent to the rectangle formed by the corresponding sides, independently of proportion and of the properties of the circle, without making use of the theorem on homothetic triangles.

BROUWER (Amsterdam): "Die möglichen Mächtigkeiten."

Seeking principally how it would be possible to construct a mathematical system, the author demonstrated that it can not be possible for more than three infinite powers to exist, viz., the enumerable, the incomplete enumerable, and the continuum.

DELITALA ( ): "La tetragonometria piana nelle scuole secondarie."

It was proposed to reunite and complete some results which the author had already published on tetragonometry. He gives the fundamental problems in the following terms. Given 5 of the 18 elements of the plane tetragon, of which at least one is a side, find the remaining 13 elements.

This contains within itself the enunciation of many problems in elementary geodesy. He asks if one should not seek to introduce by elementary methods this geometry into the secondary schools.

C. L. E. MOORE.

NOTES.

THE July number (volume 9, number 3) of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Brilliant points of curves and surfaces," by W. H. ROEVER; "Continuous increasing functions of finite and transfinite ordinals," by O. VEBLEN; "Projective differential geometry of curved surfaces (third memoir)," by E. J. WILCZYNSKI; "Invariants of the function $F(x, y, x', y')$ in the calculus of variations," by A. L. UNDERHILL; "The integration of a sequence of functions and its application to iterated integrals," by R. G. D. RICHARDSON.

THE July number (volume 30, number 3) of the *American Journal of Mathematics* contains the following papers: "Determination of conjugate points for discontinuous solutions," by O. BOLZA; "Mathematical logic as based on the theory of types," by B. RUSSELL; "Invariantive reduction of quadratic forms in the $GF[2^n]$," by L. E. DICKSON; "The motion of a particle attracted toward a fixed center by a force varying inversely as the fifth power of the distance," by W. D. MACMILLAN.

THE concluding (July) number of volume 9 of the *Annals of Mathematics* contains the following papers: "On the spherical representation of a surface," by P. SAUREL; "The absolute minimum in the problem of the surface of revolution of minimum area," by Miss M. E. SINCLAIR; "Note on the roots of Bessel functions," by C. N. MOORE; "A smooth closed curve composed of rectilinear segments with vertex points which are nowhere dense," by E. R. HEDRICK; "Evaluation of the probability integral," by F. GILMAN; "On a second theorem of the mean," by C. N. HASKINS; "Another proof of a theorem in multiply perfect members," by R. D. CARMICHAEL; "A theorem concerning equal ratios," by J. L. COOLIDGE; "Note on certain iterated and multiple integrals," by W. A. HURWITZ.

THE Italian mathematical society, Mathesis, is being fundamentally reorganized and its activities extended to assist in the reformation of the mathematical studies in the secondary schools of Italy. During the summer numerous local conventions have been held to consider the three questions: 1) scope and purpose of the society; 2) examination of the report of the commission appointed by the king to reform the secondary schools; 3) preparation of teachers of mathematics. The next general meeting will be held at Florence, October 17–22, to complete the reorganization. The names of the members of the provisional committee are Professors G. Riboni, of Milan, A. Conti, of Bologna, G. Lazzeri, of Livorno, and F. Severi, of the University of Padua.

THE Belgian academy of sciences announces the following prize subjects for 1909; the prize in each case is 800 francs:

Determine, in altitude and azimuth, the expressions for the principal terms of the periodic deviations from the vertical, under the hypothesis of the non-coincidence of the center of gravity of the crust and the center of the earth.

An important contribution to the infinitesimal geometry of ruled euclidean space is desired, with a resumé of the works already published, upon which such contribution is based.

A synopsis of the works upon systems of conics in space, with further investigations in such systems.

Manuscripts should be written in French, Flemish, or Latin, and be submitted to the secretary under the usual conditions before August 1, 1909.

THE academy of sciences of Göttingen makes the following announcement under date of June 27, 1908. According to the will of the late Dr. Paul Wolfskehl, of Darmstadt, a prize of one hundred thousand marks is offered for the proof of the greater Fermat theorem. Fermat asserts that the equation $x^\lambda + y^\lambda = z^\lambda$ cannot be satisfied by whole numbers when λ is any odd prime. The theorem is to be proved, either by Fermat processes or in completion of the investigations of Kummer (*Crelle's Journal*, volume 40, page 130; *Abhandlungen der Akademie der Wissenschaften zu Berlin*, 1857). For further literature compare Hilbert, Theorie der algebraischen Zahlkörper, *Jahresbericht der deutschen Mathematiker-Vereinigung*, volume 4 (1894), § 172, and Encyklopädie der Mathematischen Wissenschaften, Band I, Teil 2, Arithmetik und Algebra (1900–1904), IC 4b, page 713. The prize will be awarded under the following conditions:

The academy of sciences of Göttingen will determine to whom the prize shall be awarded. It will not consider any manuscript which is sent to it in competition for the prize, but will consider all memoirs which appear in the mathematical periodicals, or which are for sale in the book trade as monographs or as books. The academy requests the authors of such memoirs to send it five printed copies. Memoirs written in languages unknown to the judges will not be considered unless accompanied by authentic translations. The academy assumes no responsibility for the non-consideration of investigations to which its attention has not been called, or for mistakes that may arise because the real author of a memoir or part of a memoir is unknown to the academy. In case several persons participate in the actual solution, the academy reserves to itself the right of determining how the prize shall be divided. The award of the prize by the academy will be made not earlier than two years after the publication of the memoir for which it is awarded. During this period opportunity will be given mathematicians to express an opinion on the correctness of the solution selected, which will be made public. If the prize is awarded by the academy the successful competitors will be notified by the secretary in the name of the academy, and the action published throughout the world. The award can not be contested. The payment of the prize will be made within three months after it has been awarded, from the treasury of the University of Göttingen. Should the securities which are provided for the prize depreciate in value, the award will be considered made when they are surrendered to the successful competitor. These conditions are now in force, but will lapse September 13, 2007, if the prize is not previously awarded.

THE Göttingen academy announces also the following prize problem for 1909:

A new exact determination of a specific electron charge and its variation with varying velocity is desired. All the published results of experiments are to be carefully examined, and the various electron theories critically examined. Competing memoirs should be sent to the secretary before February 1, 1909. The prize is M. 1000.

THE royal institute of Lombardy announces the following prize subject for 1908:

The theory of transformations founded by Lie and developed during the last thirty years has been found useful in the most varied applications to geometry and mathematical analysis. It is desired to extend or complete this theory in some important aspect. Memoirs should be written in Italian, French, or Latin and forwarded to the secretary at Milan, Palazzo di Brera, before December 31, 1908.

THE Berlin academy of sciences has divided the prize of 5000 marks for the analysis of the Leibniz system of philosophy between Dr. W. KABITZ, of Breslau, and Dr. P. RITTER, of Berlin, in recognition of their work on the critical catalogue of Leibniz manuscripts.

THE philosophical faculty of the University of Berlin has awarded the prize for the solution of the problem of determining the newtonian potential of a non-homogeneous polyhedron to H. CLAUSSEN.

THE publishing house of B. G. Teubner in Leipzig dedicated the 101st edition of its catalogue of mathematical and technical books to the fourth international congress of mathematicians held in Rome last April. A portrait of Galileo appropriately appears opposite the title page. Preceding the catalogue proper is an extensive introduction giving the history of the development of the firm, which is almost synchronous with the development of mathematical literature in Germany. Besides the books which have already appeared a large number still in preparation are listed, and the plan for further extension is explained. Added to the catalogue proper is a long list of biographical notices of well known mathematicians, much more elaborate and extensive than the former list. The volume of over 600 pages is itself no small contribution to mathematical literature.

THE following courses in mathematics are announced for the winter semester of 1908–1909.

UNIVERSITY OF BERLIN.— By Professor H. A. SCHWARZ; Differential calculus, four hours: Synthetic geometry, four hours; Theory of complex numbers, two hours; Seminar, two hours; Colloquium, two hours.— By Professor G. FROBENIUS: Algebra, four hours; Analytic geometry, four hours; Seminar, two hours. —By Professor F. SCHOTTKY: General theory of functions, four hours; Theta series, two hours; Seminar, two hours.— By Professor G. HETTNER: Theory of the potential, two hours.— By Professor J. KNOBLAUCH: Theory and application of determinants, four hours; Surfaces and space curves, four hours; Proseminar, one hour.— By Professor R. LEHMANN-FILHÉS: General mechanics, four hours.— By Professor E. LANDAU: Integral calculus, four hours; Theory of aggregates, four hours.— By Dr. I. SCHUR: Theory of numbers, four hours; Elliptic functions, four hours.— By Dr. G. VALENTIN: Vector analysis, two hours.

UNIVERSITY OF BONN.— By Professor E. STUDY: Elliptic functions, three hours; Complex geometry, two hours; Seminar, one hour.— By Professor G. KOWALEWSKI: Differential and integral calculus, II, four hours; Fourier's series, two hours; Theory of aggregates, two hours; Seminar, two hours. — By Professor F. LONDON; Analytic geometry, four hours; Descriptive geometry, II, three hours.— By Dr. E. SCHMIDT: Algebra, four hours.— By Dr. C. CARATHÉODORY: Mechanics, two hours.

UNIVERSITY OF GÖTTINGEN.— By Professor F. KLEIN: Mechanics, four hours; Seminar, two hours.—By Professor D. HILBERT: Theory of functions, four hours; Principles of mathematics, two hours; Seminar, two hours.— By Professor H. MINKOWSKI: Algebra, four hours; Theory of probabilities, two hours; Seminar (with Professor Hilbert).— By Professor C. RUNGE: Differential and integral calculus, II. six hours; Seminar, two hours.— By Professor L. PRANDTL: Statics, three hours; Seminar, two hours.—By Professor W. VOIGT: Partial differential equations of mathematical physics, four hours.—By Professor E. ZERMELO: Calculus of variations, four hours. By Professor G. HERGLOTZ: Theory of orbits, two hours.—By Professor F. BERNSTEIN: History of modern mathematics, two hours; Principles of insurance, two hours; Seminar in insurance, two hours.—By Dr. P. KOEBE: Descriptive geometry with exercises, eight hours.—By Dr. O. TOEPLITZ: Theory of groups, two hours.

UNIVERSITY OF LEIPZIG.—By Professor C. NEUMANN: Differential and integral calculus, four hours.—By Professor O. HÖLDER: Mechanics, four hours; Algebra, four hours; Seminar, two hours.—By Professor K. ROHN: Differential geometry of curves and surfaces, four hours; Surfaces of the third order, two hours; Seminar, two hours.—By Professor

F. HAUSDORFF: Series and definite integrals, four hours. — By Professor H. LIEBMANN: Analytic geometry of space, four hours.

UNIVERSITY OF MUNICH. — By Professor F. LINDEMANN: Plane analytic geometry, four hours; Ordinary and partial differential equations, four hours; Foundations of insurance, two hours; Seminar, two hours. — By Professor A. VOSS: Differential calculus, four hours; with exercises, two hours; Differential geometry of curves and surfaces, I, four hours; Seminar, two hours. — By Professor A. PRINGSHEIM: Algebra, four hours; Elliptic functions, four hours. — By Professor A. SOMMERFELD: Theory of electrons, four hours; Theory of the top, two hours. — By Professor K. DOEHLEMANN: Descriptive geometry, with exercises, eight hours; Line geometry, four hours; Artistic perspective, two hours. — By Professor H. BRUNN: Theory of point sets, four hours. — By Dr. G. HARTOGS: Algebraic analysis, four hours. — By Dr. O. PERRON: Theory and application of determinants, four hours; Divergent series, two hours.

UNIVERSITY OF STRASSBURG. — By Professor H. WEBER: Differential and integral calculus, four hours; Theory of elliptic functions, two hours; Seminar, two hours. — By Professor J. WELLSTEIN: Encyclopedia of elementary mathematics, II, three hours; Linear differential equations, two hours; Seminar, one hour. — By Professor E. TIMERDING: Plane analytic geometry, three hours; with exercises, one hour; Descriptive geometry, with exercises, four hours; Spherical trigonometry, one hour. — By Professor M. SIMON: Non-euclidean geometry in elementary treatment, two hours. — By Professor P. EPSTEIN: Introduction to the theory of numbers, two hours.

PROFESSOR E. PICARD, of the University of Paris, has been elected vice-president of the academy of sciences of Paris.

PROFESSOR K. PETR, of the Bohemian University of Prague, has been promoted to a full professorship of mathematics.

PROFESSOR F. PROHAZKA, of the technical school at Brünn, has been appointed professor of geometry at the Bohemian technical school at Prague.

DR. M. FRÉCHET has been appointed master of mathematical conferences at the University of Rennes.

DR. E. ZORETTI has been appointed master of mathematical conferences at the University of Grenoble to succeed Dr. M. DULAC, who has been appointed acting professor of mathematics at the University of Poitiers.

PROFESSOR P. BURGATTI, of the University of Messina, has been appointed associate professor of mechanics at the University of Bologna.

DR. T. BOGGIO, of the University of Turin, has been appointed associate professor of mechanics at the University of Messina.

PROFESSOR G. DEFRANCHIS, of the University of Parma, and Professor M. PIERI, of the University of Catania will exchange chairs.

PROFESSOR F. GERBALDI, of the University of Palermo, has been appointed professor of algebra and analytic geometry at the University of Pavia.

PROFESSOR G. HERGLOTZ, of the University of Göttingen, has been appointed professor of mathematics at the University of Vienna.

DR. P. EPSTEIN, of the University of Strassburg, has been promoted to an associate professorship of mathematics.

PROFESSOR W. SCHLINK, of the technical school at Brunswick, has been promoted to a full professorship of theoretical mechanics.

PROFESSOR A. KNESER, of the University of Breslau, and Professor K. HENSEL, of the University of Marburg, have both declined calls to the vacant professorships of mathematics at the University of Leipzig.

PROFESSOR A. GUTZMER, of the University of Halle, has been decorated with the order of the red eagle of the fourth class.

PROFESSOR F. SCHUR, of the technical school at Karlsruhe, has been appointed professor of mathematics at the University of Strassburg, as successor to Professor TH. REYE, who has retired.

PROFESSOR H. BURKHARDT, of the University of Zürich, has been appointed professor of mathematics at the technical school at Munich, as successor to the late Professor Braunmühl.

PROFESSOR H. LAMB, of the University of Manchester, has received the honorary degree of doctor of laws from Cambridge University.

THE London mathematical society awarded the De Morgan prize for 1908 to Dr. J. W. L. GLAISHER for his researches in pure mathematics.

THE order of knighthood has been conferred upon Professor A. G. GREENHILL, of the ordnance college at Woolwich.

MR. W. F. STANTON has been appointed instructor in mathematics at University College, London.

DR. H. G. KEPPEL, of Northwestern University, has been appointed professor of mathematics at the University of Florida.

AT Indiana University Professors S. C. DAVISSON and D. A. ROTHROCK have been promoted to full professorships of mathematics; Dr. C. HASEMAN has been promoted to an assistant professorship of mathematics.

PROFESSOR J. E. SINCLAIR, of the Worcester Polytechnic Institute, has retired from active service.

MR. J. W. MILLER and Mr. J. E. STOCKER have been promoted to assistant professorships of mathematics at Lehigh University.

THE following changes have taken place at the Massachusetts Institute of Technology: Dr. W. H. ROEVER has been appointed assistant professor of mathematics at Washington University, St. Louis; Mr. J. N. LENNES has been appointed assistant professor of mathematics at Brown University; Dr. C. L. E. MOORE has returned from his leave of absence in Italy and will resume his duties as instructor of mathematics; Mr. E. A. MILLER has retired to enter business; Mr. J. LIPKE, of the University of California, has been appointed instructor in mathematics.

MR. H. H. DALAKER has been appointed assistant professor of mathematics at the University of Minnesota.

MR. E. P. R. DUVAL, of the University of Wisconsin, has been appointed associate professor of mathematics at the University of Oklahoma.

DR. FRANK IRWIN has been appointed instructor in mathematics at Princeton University.

DR. G. M. COLLWELL has been appointed instructor in mathematics at Yale University.

PROFESSOR C. E. BIKLÉ, of Teachers College, Columbia University, has resigned to accept a position in the public schools at Syracuse, N. Y.

MR. A. B. FRIZELL, of Harvard University, has been appointed professor of mathematics and astronomy in Midland College, Atchison, Kansas.

PROFESSOR F. W. HANAWALT, of Albion College, has been appointed professor of mathematics and astronomy at the University of Puget Sound.

PROFESSOR FLOYD FIELD, of the Georgia School of Technology, has been appointed acting head of the department of mathematics in the absence of Professor O. T. GECKELER, who will spend the coming year in study at the University of Pennsylvania.

THE death is announced of ISRAEL C. PIERSON, for thirty-five years actuary of the Washington Life Insurance Company of New York. Mr. Pierson had been a member of the AMERICAN MATHEMATICAL SOCIETY since 1889.

PROFESSOR H. JOLY, of the University of Lausanne, died August 4, at the age of 48 years.

DR. CHARLES TAYLOR, Master of St. John's College, Cambridge, died August 12, 1908, at the age of 43 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ALCAYDE (N.). Cálculo de probabilidades. Obra declarada de texto para la Academia de ingenieros del ejército por real orden de 23 de Octubre de 1907. Guadalajara, Colegio de Huérfanos de la Guerra, 1908. 8vo. 79 pp.

BOLZA (O.). Vorlesungen über Variationsrechnung. Umgearbeitete und stark vermehrte deutsche Ausgabe der "Lectures on the calculus of variations." In drei Lieferungen. Lieferung I. Leipzig, Teubner, 1908. 8vo. 4 + 300 pp. M. 8.00

BURKHARDT (H.). Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik. Bericht, erstattet der deutschen Mathematiker-Vereinigung. (In zwei Halbbänden.) Leipzig, Teubner, 1908. 8vo. 12 + 3 + 1804 pp. M. 30.00

CANTOR (M.). Vorlesungen über Geschichte der Mathematik. Band IV. 5te Lieferung, Abschnitt 27: Wallner, C. R., Totale und partielle Differentialgleichungen; Differenzen- und Summenrechnung; Variationsrechnung. Abschnitt 28: Cantor, M., Ueberblick über die Zeit von 1758 bis 1799. Leipzig, Teubner, 1908. 8vo. Pp. 883–1113. M. 6.80

DÜSING (K.). Die Elemente der Differential- und Integralrechnung in geometrischer Methode dargestellt. Ausgabe A. Für Gymnasien, Realgymnasien und Oberrealschulen, sowie zum Selbstunterricht. Mit zahlreichen Uebungen. Hannover, Jäneke, 1908. 8vo. 74 pp. Cloth. M. 1.30

EGERER (H.). Repetitorium der höheren Mathematik. (Lehrsätze, Formeln, Tabellen.) München, Oldenbourg, 1908. 8vo. 8 + 351 pp. M. 6.00

EMMERICH (A.). Pseudogleichschenklige Dreiecke im Bereich der Winkelhalbirenden. Teil I. Mühlheim, 1908. 4to. 32 pp.

FICK (E.). Anfangsgründe der analytischen Geometrie des 4-fach ausgedehnten Raumes. (Progr.) Neuburg, 1907. 8vo. 55 pp.

FRATTINI (G.). La nozione d'indice e l'analisi indeterminata dei polinomi interi: lettura al IV Congresso internazionale dei matematici. Roma, Bertero, 1908. 16mo. 7 pp.

GALOIS (E.). See TANNERY (J.).

GEIGER (K.). Zwei Kurven zweiter Ordnung in zwei- und vierpunktiger Berührung. Eine synthetischgeometrische Studie. (Progr.) Landshut, 1907. 8vo. 37 pp.

HAWLITSCHEK (K.). Integration irrationaler Differentiale. (Progr.) Prag-Neustadt, 1907. 8vo. 15 pp.

HIESSL (M.). Einige Elemente der vierdimensionalen Geometrie. Teil II. (Progr.) Stockerau, 1907. 8vo. 17 pp.

HILB (E.). Ueber Integraldarstellungen willkürlicher Funktionen. Erlangen, 1908. 8vo. 66 pp.

ILNICKI (E.). Ueber die Prinzipien der Infinitesimalrechnung und über die Wandlungen, welche die Darstellung dieses Zweiges der Mathematik im Laufe seiner Entwickelung erfahren hat. (Schluss.) (Progr.) Czernowitz, 1907. 8vo. 9 pp.

JOHN (F.). Ueber die kürzesten Linien auf krummen Oberflächen im allgemeinen und über die auf dem dreifachen Ellipsoide im besonderen. Teil I. (Progr.) Leitmeritz, 1907. 8vo. 21 pp.

JUNKER (F.). Höhere Analysis. Teil II. Integralrechnung. 3te verbesserte Auflage. (Sammlung Göschen, 88.) Leipzig, Göschen, 1908. 12mo. 190 pp. Cloth. M. 0.80

KLUGE (W.). Besondere Systeme. Ein Beitrag zur Bestimmung von Determinanten. (Progr.) Lissa, Eulitz, 1908. 8vo. 48 pp. M. 1.50

LENZI (F.). Primi elementi di calcolo differenziale e integrale. Livorno, Giusti, 1908. 16mo. 8 + 92 pp. L. 1.00

MONTESSUS (R. DE). Leçons élémentaires sur le calcul des probabilités. Philosophie du hasard, principes du calcul des probabilités, jeux de hasard, jeux savants, la spéculation, probabilité géométrique, probabilité des causes, etc. Paris, Gauthier-Villars, 1908. 8vo. 6 + 192 pp. F. 7.00

Nordio (A.). Dell'inviluppo dei piani comuni a due quadriche. (Quadrispirale sviluppabile.) (Progr.) Triest, 1907. 8vo. 72 pp.

Pölt (J.). Bestimmung der Typen derjenigen viergliedrigen Gruppen in beliebig vielen Veränderlichen, die keine dreigliedrige Involutions-Untergruppe besitzen. (Progr.) Innsbruck, 1907. 8vo. 28 pp.

Prang (C.). Determinanten. I. Hauptsätze über Determinanten. II. Einleitung in die analytische Geometrie des Raumes unter Anwendung der Determinanten. 2te Auflage. Berlin, Mayer und Müller, 1908. 8vo. 6 + 65 pp. Boards. M. 2.00

Reim (H.). Das regelmässige Dodekaeder und Ikosaeder in ihren wechselseitigen Beziehungen nach Angaben von L. Huebner. (Progr.) Schweidnitz, 1907. 4to. 28 pp.

Reisenhofer (R.). Das Apollonische Berührungsproblem in stereographischer Projektion. (Progr.) Bielitz, 1907. 8vo. 9 pp.

Schwefel (A.). Betrachtungen über imaginäre Kreise. Versuch einer anderen Fassung der Möbiusschen Abhandlung. (Progr.) Brünn, 1907. 8vo. 10 pp.

Sibiriani (F.). Formulario di matematiche ad uso degli studenti universitari. Bologna, 1908. 12mo. 350 pp. M. 4.50

Sturm (R.). Lehre von den geometrischen Verwandtschaften. (4 vols.) Vol. I: Verwandtschaften zwischen Gebilden erster Stufe. Leipzig, 1908. 8vo. 12 + 415 pp. Cloth. M. 16.00

Tannery (J.). Manuscrits de E. Galois. Paris, 1908. 8vo. 69 pp. M. 2.50

Wallner (C. R.). See Cantor (M.).

Weber (H.). Lehrbuch der Algebra. 2te Auflage. Band III: Elliptische Funktionen und algebraische Zahlen. Braunschweig, Vieweg, 1908. 8vo. 14 + 733 pp. M. 20.00

Wirth (H.). Beiträge zur Theorie der Abbildungen durch reciproke radii vectores. (Progr.) Wolgast, 1907. 8vo. 32 pp.

II. ELEMENTARY MATHEMATICS.

Bortolotti (E.). Nozioni pratiche di geometria e rudimenti di disegno geometrico. 4a edizione riveduta e corretta. Roma-Milano, Segati, 1907. 8vo. 128 pp. L. 1.50

Bourlet (C.). Éléments de géométrie (géométrie plane, géométrie dans l'espace) contenant 762 exercises, rédigés conformément aux programmes de l'enseignement secondaire des garçons et de l'enseignement secondaire des jeunes filles. Paris, Hachette, 1908. 16mo. 6 + 382 pp. F. 2.50

Bouvart (C.) et Ratinet (A.). Nouvelles tables de logarithmes à cinq décimales. Table numérique. Tables trigonométriques. Paris, Hachette, 1908. 8vo. 176 pp. F. 2.50

Bullard (W. G.) See Metzler (W. H.).

Cannaviello (M.). Corso di geometria elementare per le scuole medie. Parte I: Planimetria. Napoli, Priore, 1908. 8vo. 322 pp. L. 2.00

Castle (F.). Logarithmic and other tables for schools. New York, Macmillan, 1908. 12mo. 36 pp. $0.20

Chini (M.). Lezioni di algebra ad uso dei licei. Vol. II. Livorno, Giusti, 1908. 16mo. 9 + 172 pp. L. 1.80

Cortázar (D.). See Cortázar (J.).

Cortázar (J.). Tratado de geometria elemental. 35a edición, corregida y arreglada por D. Cortázar. Madrid, Tello, 1908. 8vo. 207 pp. P. 4.00

Deakin (R.). Key to new geometry papers. London, Macmillan, 1908. 12mo. 3s. 6d.

Gremigni (M.). Elementi di geometria ad uso delle scuole tecniche e professionali. Planimetria, libro III e IV. 3a edizione. Firenze, Bemporad, 1908. 16mo. 146 pp. L. 0.90

Haase (E.). See Wilk (E.).

Hammer (E.). Der logarithmische Rechenschieber und sein Gebrauch. Eine elementare Anleitung zur Verwendung des Instruments fur Studierende und für Praktiker. 4te durchgesehene Auflage. Stuttgart, Wittwer, 1908. 8vo. 8 + 80 pp. M. 1.00

Hartl (H.). Uebungsbuch für den Unterricht in der allgemeinen Arithmetik und Algebra an Werkmeisterschulen, Baugewerkenschulen und verwandten Lehranstalten. Ausgegeben für die Schulen im Deutschen Reiche. 2te Auflage. Wien, Deuticke, 1908. 8vo. 6 + 161 pp. M. 2.00

Jacob (J.). Lehrbuch der Arithmetik für Obergymnasien. Wien, Deuticke, 1908. 8vo. 6 + 292 pp. M. 3.20

Laisant (C. A.). Einführung in die Mathematik. Allen Kinderfreunden gewidmet. Deutsch von F. J. Schicht. Wien, Deuticke, 1908. 8vo. 15 + 199 pp. M. 2.00

Lalande (J. de). Tables de logarithmes, étendues à sept décimales. Nouvelle édition, augmentée de formules pour la résolution des triangles. Paris, Gauthier-Villars, 1907. 16mo. 42 + 236 pp. F. 3.50

Launay (L.). Calcul de géométrie intuitive. 9e édition. Paris, Delagrave, 1907. 18mo. 251 pp.

Lesser (O.). Graphische Darstellungen im Mathematikunterricht der hoheren Schulen. Eine Sammlung von Materialien für die Hand des Lehrers. Leipzig, Freytag, 1908. 8vo. 108 pp. M. 5.00

Martin (E.). Arithmétique, système métrique, notions de géométrie. Paris, Delagrave, 1908. 16mo. 16 + 356 pp. F. 3.00

Martin Mengod (A.). Prolegómenos de matemáticas (nociones de aritmética y geometria). Orense, La Popular, 1907. 8vo. 176 pp. P. 6.50

Metzig (C.). Lehrbuch der Arithmetik und Algebra, verbunden mit einer Aufgabensammlung von mehr als 5800 Aufgaben für Baugewerkschulen, Maschinenbauschulen und andere technische Lehranstalten, für Realschulen und Mittelschulen sowie zum Selbstunterrichte. 3te vermehrte und verbesserte Auflage. Breslau, Morgenstern, 1908. 8vo. 188 pp. Cloth. M. 2.40

Metzler (W. H.), Roe (E. D.) and Bullard (W. G.). College algebra. New York, Longmans, Green and Co., 1908. 352 pp. $1.50

Müller (H.). Die Mathematik auf den Gymnasien und Realschulen. Für den Unterricht dargestellt. Teil I. Die Unterstufe. (Lehraufgabe der Klassen Quarta bis Unter-Sekunda.) 4te Auflage. Ausgabe A. Für Gymnasien und Progymnasien. Leipzig, Teubner, 1907. 8vo. 8 + 136 pp. Cloth. M. 1.60

NELL (A. M.). Fünfstellige Logarithmen der Zahlen und der trigonometrischen Functionen, nebst den Logarithmen für Summe und Differenz zweier Zahlen, deren Logarithmen gegeben sind, sowie einigen anderen Tafeln. 12te Auflage. Giessen, Roth, 1908. 20 + 104 pp. Cloth. M. 2.00

PISATI (L.). Elementi di geometria ad uso delle scuole medie inferiori. Torino, Paravia, 1907. 8vo. 5 + 202 pp. L. 2.40

RATINET (A.). See BOUVART (C.).

RICHTER (O.). Kreis und Kugel in senkrechter Projektion. Für den Unterricht und zum Selbststudium. Leipzig, Teubner, 1908. 8vo. 10 + 188 pp. Cloth. M. 4.80

ROE (E. D.). See METZLER (W. H.).

SCHUBERT (H.). Vierstellige Tafeln und Gegentafeln für logarithmisches und trigonometrisches Rechnen, in zwei Farben zusammengestellt. 3te verbesserte Auflage. (Sammlung-Göschen, 31.) Leipzig, Göschen, 1908. 16mo. 128 pp. Cloth. M. 0.80

SERRET (J. A.). Traité de trigonométrie. 9e édition. Paris, Gauthier-Villars, 1908. 8vo. 10 + 336 pp. F. 4.00

THIEME (H.). Leitfaden der Mathematik für Gymnasien. 2 Teile. Leipzig, Freytag, 1908. 8vo. 214 pp. M. 3.20

VAN DER HAYDEN (A. F). Algebraic examples. Book I. London, Arnold, 1908. 8vo. 96 pp. 1s.

VILLETTE (T.). Eléments de géométrie pratique et de dessin linéaire, figures planes, figures dans l'espace, applications, dessin à main levée, dessin géométrique, lavis, 215 figures, 550 problèmes. Cours moyen et supérieur, rédigé conformément au programme de géométrie des écoles de Paris. 7e édition. Paris, Larousse, 1908. 8vo. 2 + 172 pp.

WELLS (W.). New plane and solid geometry. Boston, Heath, 1908. 12mo. 7 + 120 pp. Cloth. $1.25

WILK (E.) und HAASE (E.). Anweisung zur Geometrie der Mittelschule. Teil II. Dresden, Bleyl und Kaemmerer, 1908. 8vo. 8 + 104 pp. M. 1.80

III. APPLIED MATHEMATICS.

AMTMANN (H.). Neue mathematische Theorie der Witwenversicherung. Jena, 1908. 8vo. 56 pp. M. 2.50

BLAKE (C. F.). Mathematics of machine design with special reference to shafting and efficiency of hoisting machinery. New York, Industrial Press, 1908. 8vo. 35 pp. $0.25

CREW (H.). The principles of mechanics; for students of physics and engineering. New York, Longmans, 1908. 8vo. 10 + 295 pp. Cloth. $1.50

HENNELL (T.). Hydraulic and other tables. 3d edition. London, Spon, 1908. 8vo. 4s. 6d.

ILNICKI (E.). Maturitätsaufgaben aus der darstellenden Geometrie nebst vollständigen Losungen für die oberen Klassen der Realschulen. Wien, Deuticke, 1908. 8vo. 3 + 82 pp. M. 1.25

JOUGET (E.). Lectures de mécanique. La mécanique enseignée par les auteurs originaux. Part I: La naissance de la mécanique. Paris, Gauthier-Villars, 1908. 8vo. 10 + 210 pp. F. 5.00

Kohnstamm (P.). See Waals (J. D. van der).

Küster (F. W.). Logarithmische Rechentafeln für Chemiker, Pharmazeuten, Mediziner und Physiker. Im Einverständnis mit der Atomgewichtskommission der deutschen chemischen Gesellschaft für den Gebrauch im Unterrichtslaboratorium und in der Praxis berechnet. 8te Auflage. Leipzig, Veit, 1908. 12mo. 107 pp. Cloth. M. 2.40

Mach (E.). Die Mechanik in ihrer Entwickelung historisch-kritisch dargestellt. 6te verbesserte und vermehrte Auflage. Leipzig, Brockhaus, 1908. 8vo. 18 + 576 pp. Cloth. M. 9 00

Mayer (J. E.). Das mechanische Rechnen des Ingenieurs (Rechenschieber, Rechenmaschinen, Planimeter, Integrator, Integraph). Hannover, Janecke, 1908. 12mo. Cloth. M. 2.20

Newcomb (S.). La théorie du mouvement de la lune: son histoire et son état actuel. Roma, L'Union coopérative éditrice, 1908. 8vo. 10 pp.

Orlandi (G.). Tavola grafica tacheopantometrica per determinare colla stessa precisione delle tavole numeriche e con maggior rapidità le distanze orizzontali, le differrenze di livello, le coordinate planimetriche e le curve: instruzioni sul modo d'usare la tavola. Roma, Instituto geografico De Agostini, 1908. 16mo. 15 pp.

Richard (G.). Mécanique revue. A l'usage des ingénieurs, constructeurs-mécaniciens, industriels, chefs d'ateliers et contremaîtres. 30e édition. Paris, Dunod et Pinat, 1908. 16mo. 6 + 210 pp. F. 2.50

Ruiz Castizo (J.). Tratado de mecánica racional. Vol. I: Teoria general de los sistemas de vectores; cinemática. Madrid, Victoria-Suárez, 1908. 8vo. 15 + 582 pp. P. 18.00

Salmoiraghi (A.). Instrumenti e metodi moderni di geometria applicata. Parte I: Teoria degli instrumenti misuratori, descrizione e norme pratiche per l'uso. Milano, 1907. 8vo. 23 + 495 pp.

Sarrette (H.). Précis arithmétique des calculs d'emprunts à long terme et de valeurs mobilières. Paris, Gauthier-Villars, 1908. 8vo. 19 + 292 pp. F. 10.00

Waals (J. D. van der). Lehrbuch der Thermodynamik in ihrer Anwendung auf das Gleichgewicht von Systemen mit gasformig-flüssigen Phasen. Nach Vorlesungen bearbeitet von P. Kohnstamm. Teil I. Leipzig, Mass und Van Suchtelen, 1908. 8vo. 12 + 287 pp. Cloth. M. 12.00

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THE FIFTEENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE fifteenth summer meeting of the Society was convened at the University of Illinois on Thursday and Friday, September 10–11, 1908. The scientific proceedings were completed in four full sessions, on Thursday and Friday mornings and afternoons. The social gathering and dinner on Thursday evening at the University Club afforded much pleasure and satisfaction to the members present.

The first session was opened with an address of welcome by Professor Townsend on behalf of the University of Illinois. Owing to the absence of Professor White, President of the Society, all sessions were presided over by Vice-President Professor G. A. Miller, except on two occasions while Professor Miller was presenting his own papers, when Professors Van Vleck and Ziwet respectively were called to the chair. At the close of the Friday morning session resolutions were adopted expressing the Society's appreciation of the generous hospitality of the University of Illinois and its officers.

The total attendance at the various sessions was over fifty, including the following thirty-eight members of the Society:

Professor L. D. Ames, Professor G. A. Bliss, Dr. R. L. Börger, Professor W. D. Cairns, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor S. C. Davisson, Dr. E. L. Dodd, Professor L. W. Dowling, Dr. O. Dunkel, Professor C. Haseman, Professor E. R. Hedrick, Mr. T. H. Hildebrandt, Professor C. N. Haskins, Dr. L. Ingold, Professor O. D. Kellogg, Professor H. W. Kuhn, Dr. E. B. Lytle, Dr. W. R. Longley, Professor G. A. Miller, Professor Max Mason, Dr. C. N. Moore, Professor H. B. Newson, Dr. L. T. Neikirk, Mr. E. W. Ponzer, Professor W. H. Roever, Professor H. L. Rietz, Professor D. A. Rothrock, Professor J. B. Shaw, Professor H. E. Slaught, Mr. L. L. Silverman, Professor E. J. Townsend, Professor J. H. Tanner, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor J. W. Young, Professor A. Ziwet. Professor M. Abraham, of the University of Göttingen, Germany, was also present.

At the meeting of the Council on Thursday evening, the following persons were elected to membership in the Society: Dr.

G. G. Chambers, University of Pennsylvania; Dr. G. M. Conwell, Yale University; Mr. F. F. Decker, Syracuse University; Mr. W. F. Ewing, California Polytechnic School; Professor A. E. Haynes, University of Minnesota; Mr. P. H. Linehan, College of the City of New York; Mr. H. F. MacNeish, University High School, Chicago; Mr. Lewis Omer, High School, Oak Park, Ill.; Professor J. C. Stone, State Normal College, Ypsilanti, Mich. Six applications for membership in the Society were received. The total membership is now 598.

The following papers were read at the summer meeting:

(1) Professor EDUARD STUDY: "Zur Differentialgeometrie der analytischen Curven."

(2) Professor L. E. DICKSON: "On definite forms in a finite field."

(3) Professor G. A. MILLER: "Answer to a question raised by Cayley as regards a property of abstract groups."

(4) Mr. ARNOLD DRESDEN: "The second derivatives of the extremal integral."

(5) Dr. O. E. GLENN: "On the decomposition of forms into quadratic factors."

(6) Dr. LOUIS INGOLD: "On the Kowalewski integral."

(7) Professor L. D. AMES: "A method for the approximate solution of n equations in n unknowns."

(8) Professor OSKAR BOLZA: "Heinrich Maschke: his life and work."

(9) Professors G. A. BLISS and MAX MASON: "Fields of extremals in space."

(10) Professor J. B. SHAW: "Qualitative algebra."

(11) Professor VIRGIL SNYDER: "Construction of plane curves of given order and genus having distinct double points."

(12) Dr. ARTHUR RANUM: "The divisions of riemannian space into congruent parts."

(13) Dr. ARTHUR RANUM: "Parallelopipeds in lobachevskian geometry."

(14) Dr. F. L. GRIFFIN: "Families of central orbits related to circular trajectories."

(15) Dr. F. R. SHARPE: "The identical relations of the strain and stress components of an elastic solid."

(16) Professor VIRGIL SNYDER: "Surfaces derived from the cubic variety having nine double points in four dimensional space."

(17) Professor L. E. DICKSON: "On Fermat's last theorem."

(18) Professor L. E. DICKSON: "Rational reduction of two quadratic forms" (preliminary communication).

(19) Dr. W. R. LONGLEY: "Note on implicit functions."

(20) Professor E. B. WILSON: "Note on statistical mechanics."

(21) Professor E. J. WILCZYNSKI: "A projective generalization of Meusnier's theorem."

(22) Professor E. R. HEDRICK: "On the convergence of the jacobian."

(23) Professor EDWARD KASNER: "Conformality in connection with functions of two complex variables."

(24) Professor G. A. MILLER: "Groups with regard to modular systems."

(25) Professor J. W. YOUNG: "Two-dimensional chains and the classification of complex collineations in a plane (second paper)."

(26) Mr. J. H. MACLAGAN-WEDDERBURN: "On the direct product in the theory of finite groups."

(27) Professor W. B. FITE: "The class of a group all of whose operations except identity are of order three."

(28) Dr. C. N. MOORE: "The summability of the developments in Bessel functions, with applications."

(29) Professor E. J. TOWNSEND: "Interchange of order of differentiation."

(30) Professor H. B. NEWSON: "On characteristic equations."

(31) Professor L. W. DOWLING: "The arrangement of the real branches of a plane sextic curve."

(32) Professor C. N. HASKINS: "On the second law of the mean."

(33) Dr. L. I. HEWES: "Necessary and sufficient conditions that an ordinary differential equation of the first order and nth degree shall admit a continuous conformal group."

In the absence of the authors, the papers of Professor Study, Professor Dickson, Mr. Dresden, Dr. Glenn, Professor Snyder, Dr. Ranum, Dr. Griffin, Dr. Sharpe, Professor Wilson, Professor Kasner, Mr. Maclagan-Wedderburn, and Professor Fite were read by title. Professor Bolza's memorial of Professor Maschke was read by Professor Bliss. The papers of Professor Dickson and Professor Fite were commented upon by Professor

Miller, and the main results of Mr. Dresden's paper were presented by Professor Bliss.

Professor Snyder's first paper appeared in the October BULLETIN. Professor Miller's first paper and Professor Bolza's memorial of Professor Maschke are included in the present number of the BULLETIN. Professor Study's paper will appear in the *Transactions*. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

2. Defining a definite form to be one which represents only squares in a given field, Professor Dickson determines all m-ary definite forms of degree < 8 in each Galois field of order p^n, $p > 2$. It is shown that every definite quartic form is formally a perfect square. A like result holds for definite sextic forms when $p^n \geqq 13$; but for $p^n \leqq 11$ there occur additional types. The paper has been offered to the *Transactions*. The simplicity of the modular types is in marked contrast to results obtained in the algebraic theory by Hilbert, *Acta Mathematica*, volume 17, page 169.

4. In Mr. Dresden's paper the extremal integral is used for the derivation of necessary conditions for a minimum of the integral

$$\int_{t_1}^{t_2} F(x, y, x', y')dt$$

in the case of one or two variable endpoints and for the case of "discontinuous solutions." The second derivatives of the extremal integral are computed in order to discuss the second variation of the definite integral in these different cases. The results for the cases of one or two variable endpoints are identical with those previously obtained by Bliss. In the problem of the "discontinuous solutions" the conclusion is, in appearance, directly in contradiction with results established by Carathéodory and Bolza. This contradiction is however shown to exist in appearance only, by means of a relation between Weierstrass's E-function and Carathéodory's invariant Ω, a relation which furthermore makes possible a simple proof of a theorem previously proved by the last-named author.

5. The decomposition of ternary and quaternary forms into linear factors has been studied by Junker incidentally in a me-

moir on symmetric functions in the *Mathematische Annalen*, volume 45. In Dr. Glenn's paper the methods of Junker are extended and supplemented to construct the theory of decomposition of forms into quadratic factors. The first part of the paper is devoted to the relations which exist among the coefficients of a form in order that it degenerate into such factors. In the second part the original form is assumed to be degenerate, and the component quadratic forms completely determined. The paper forms a new section of an extended investigation on degenerate curves and surfaces, previously communicated to the Society.

6. G. Kowalewski, in his paper "Ueber den zweiten Mittelwertsatz der Integralrechnung" * has introduced a generalization of the Riemann integral. The notion involves two functions $u(x)$, $v(x)$ defined in an interval $a-b$. This interval is divided into subintervals by the points $x_0 = a, x_1, x_2, \cdots, x_n = b$. The sum $\sum_1^u (v(x_i) - v(x_{i-1}))u(\xi_i)$ is formed, where ξ_i is any value of x in the ith interval. If this sum approaches a unique limit when u increases indefinitely in such a way that the greatest subinterval approaches zero, this limit is denoted by $\int_a^b u dv$. This clearly coincides with $\int_a^b uv' dx$ whenever the latter has any meaning.

In the article referred to it is shown that $\int_a^b u dv$, in the above sense, exists, 1) if u is continuous and v is of limited variation; 2) if u is integrable and v is a definite integral function.

The principal results of Mr. Ingold's paper follow:

(1) A necessary and sufficient condition for the existence of the Kowalewski integral is that it be possible to make $\Sigma\omega_i u|\Delta_i v|$ arbitrarily small by making the largest subinterval small enough, $\omega_i u$ being the oscillation of u in the ith interval and $\Delta_i v$ standing for $v(x_i) - v(x_{i-1})$.

(2) If u is integrable, it is sufficient for the existence of $\int_a^b u dv$ that v be continuous and $\Delta v/\Delta x$ be bounded. This last condition on v, however, may be waived in neighborhoods about points of a reducible set.

(3) The elementary formulas for integration hold when the variable is replaced by a continuous function if the integral is then regarded as a Kowalewski integral.

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* *Math. Annalen*, vol. 60, pp. 151–156.

7. Professor Ames gives a method for the approximate solution of n equations in n unknowns under very broad restrictions on the functions involved. The method is a generalization of Newton's method in one unknown and is based on Taylor's theorem. A rough guess is made as to the solution, and successive approximations are obtained after the manner of Goursat's proof for the existence of implicit functions. A geometric interpretation is suggested in the case of two equations. A convergence test analogous to Goursat's is obtained, which can be applied after a sufficient number of approximations have been made, and which yields a limit of error for the root. While the work is analogous to Goursat's, and can be put in much the same form, the problem is essentially different, in that he assumes the existence of one point which satisfies the given equations, and that is what is sought in the present case. It is also found more convenient to handle the equations directly along the lines of Newton's method rather than to make Goursat's somewhat arbitrary transformation. The process frequently converges rapidly even though the first guess is far from a root. If the equations are algebraic the method can be carried out by synthetic division in a form which is a precise generalization of Horner's process, though this is probably not advisable in general.

9. The minimizing space curves for the integral

$$J = \int F(x, y, z, x', y', z')dt$$

form a four-parameter family. Of special interest are the two-parameter families which pass through a given point, or are cut transversely by a given curve or surface. Such families occur when the integral J is to be minimized with respect to curves which join two fixed points, or which have one end point fixed while the other is free to vary on a fixed curve or surface. In the study of the sufficient conditions which insure a minimum for the integral J under these conditions, it is essential to know that such two-parameter families fill out simply a portion of space about the point through which they all pass, or about the curve or surface by which they are cut transversely. The minimizing curves, the extremals, are said to form a field in the region for which this property holds. The field proofs for the analogous cases in the plane have already been made. (See Bolza, Vorlesungen über Variationsrechnung, page 249; or Lectures on the calculus of variations, page 175; Bliss, "The construction of a field of extremals about a given point," BULLETIN, volume 13 (1906), page 321. See also Bolza, "Existence proof for a field of extremals tangent to a given curve," *Transactions of the American Mathematical Society*, volume 8 (1907), page 399.) It is shown in the paper of Professors Bliss and Mason that the extremals through a fixed point, or the extremals to which a given curve is transversal, form a field. In these cases the functional determinant of the equations to be solved vanishes at the point, or along the curve in question. By a transformation of the variables the equations are reduced to a set whose functional determinant does not vanish, and the solution is then obtained by the aid of the theory of implicit functions.

10. Professor Shaw's paper deals primarily with the various forms of products in qualitative algebra, by which term is meant general algebra, universal algebra, multiple algebra, and the like. The different kinds of products are classified and examined in turn. Various formulas are deduced for the more important forms.

12. Corresponding to every finite discontinuous group of movements of order m in euclidean or non-euclidean geometry, there is a division of space into m congruent parts. In the case of riemannian space the groups were determined by Goursat and found to be of fifty-one distinct types. In the present paper Dr. Ranum derives the corresponding divisions of space. The boundaries between them are taken to be Clifford surfaces of radius $\frac{1}{4}\pi$. Solids having many of the properties of euclidean parallelopipeds play an important rôle. Study's remarkable correspondence between straight lines of riemannian space and pairs of points lying on a pair of spheres in euclidean space is utilized to great advantage.

13. In his second paper Dr. Ranum gives a classification of most important solids in lobachevskian space whose properties are analogous to those of parallelopipeds in euclidean space and which degenerate into parallelopipeds when the space becomes euclidean.

14. Dr. Griffin considers the circle as a central orbit, its center not coinciding with the center of force, and shows that

all families of such orbits are characterized by two invariant properties. He then discusses the question of the most general laws of central force admitting families of trajectories possessing one or the other of these properties; and, in particular, points out for the more usual laws the families in question.

15. The 6 components of strain of an elastic solid are linear functions of the derivatives of the 3 displacements. Six identical relations between the components were first given by St. Venant. Dr. Sharpe shows that there are 15 such relations, of which only 3 are independent. The 6 components of stress are linear functions of the 6 components of strain and are therefore subject to 15 similar relations. In the case of equilibrium under surface tractions alone there are 3 additional equations of equilibrium. Six identical relations between the stress components were first given by Beltrami. It is shown in this paper that the number of relations between any of the 6 components is equal to their number and that all but one of these are independent.

16. In the *Turin Memoirs* for 1888 Professor Segre studies a large number of forms of cubic varieties in four dimensional spaces. He also discusses the apparent contour in ordinary space, and various congruences having the contour for complete focal surface. The treatment is entirely synthetic, and no case is treated in detail. Professor Snyder considers the case of the cubic variety having nine double points from the analytic standpoint, and derives two new forms of (3, 3) congruences. The first has a focal surface with two plane cubic cuspidal curves and three triple planes, each containing six double points. The other case presents a family of congruences all contained in a fixed linear complex. The focal surface of order and class six is complete focal surface for six such congruences. It has a space sextic for cuspidal curve, nine double points, and nine double planes. Through each point pass five planes and in each plane lie five points.

17. Fermat's equation $x^n + y^n + z^n = 0$ is proved to be impossible in integers prime to n for every odd prime $n < 7000$. The method is based upon the corresponding congruence modulo p, where p is a prime of the form $mn + 1$. For each $m < 74$ and for $m = 76$ and 128, all the primes p are determined for which the congruence has solutions. The results are

given in two papers now appearing in the *Messenger of Mathematics*. In a subsequent paper, offered to *Crelle's Journal*, Professor Dickson establishes by means of the cyclotomic theory the theorem that, when m exceeds a certain cubic function of n, the congruence has a set of solutions prime to p. This theorem is in contrast to the known result, admitting of immediate proof, that, for a fixed value of m prime to 3, the congruence has no solutions except when n has one of a finite set of values.

19. In the case of one equation $f(x, t) = 0$ (where f is expansible in positive integral powers of x and t and $f(0, 0) = 0$), defining a function x of one independent variable t, the condition that the jacobian shall be different from zero (for $x = t = 0$) is a sufficient condition that the function x shall be defined uniquely as a series in positive powers of t vanishing with t. If $f(x, t)$ does not contain t as a factor, the condition is also necessary for a unique solution.

In the case of two equations $f(x, y, t) = 0$, $g(x, y, t) = 0$ (where f and g are expansible in positive integral powers of x, y, t and $f(0, 0, 0) = 0$, $g(0, 0, 0) = 0$), defining x and y in terms of t, the condition that the jacobian shall be different from zero (for $x = y = t = 0$) is a sufficient condition that x and y shall be defined uniquely as series in positive powers of t vanishing with t. Supposing that neither $f(x, y, t)$ nor $g(x, y, t)$ contains t as a factor, the condition is however not necessary for a unique solution. It is shown in Dr. Longley's note that the condition that the jacobian shall be different from zero is a special case of more general conditions upon f and g in order that x and y shall be defined uniquely as series in positive powers of t vanishing with t. For certain special cases in which the jacobian vanishes, the general conditions furnish practical criteria which are analogous to the ordinary jacobian condition. The general method can be extended to any number of equations.

20. Professor Wilson discusses the analogy between hydromechanics and statistical mechanics. In case the number of degrees of freedom of the dynamical system which is to be treated statistically is $n = 1$, tolerably satisfactory results are obtainable; when $n > 1$ the equations of the fictitious fluid which represents the statistical problem must in general be more complicated than the hydromechanic type. A discussion of the physical dimensions of the quantities arising in connec-

tion with the analogy discourages the attempt to find a medium which shall satisfactorily represent the problem.

21. Meusnier's theorem may be stated as follow: Consider those sections of a surface made by planes which pass through a fixed point and a fixed tangent of the surface. The circles which osculate these plane sections at the fixed point common to all of them generate a sphere.

In Professsor Wilczynski's generalization osculating conics, *i. e.*, conics having fourth order contact with the plane sections, take the place of the osculating circles. Their locus is a quadric surface.

The detailed discussion of special and exceptional cases gives rise to some interesting results. The author also considers the surface which is the locus of the osculating conics of those plane sections which contain the directrix of the second kind of the surface point considered. If we think of this directrix as a projective substitute for the normal, this consideration may be characterized as a projective generalization of Euler's theorem.

22. Criteria for differentiation of a series of functions of a real variable term by term have been a subject of serious study and are now well known in several forms of varying sharpness. In general they require that the series which results upon attempted differentiation should converge uniformly, with a further condition upon the original series which essentially requires its uniform convergence, though this is redundant with the previous requirement and may be partially suppressed in the statement of the theorem.

Similar criteria for the term by term differentiability of series of functions of a complex variable are usually stated, but the conditions are apparently smaller on account of the usual assumption that the function is analytic.

Professor Hedrick's paper considers in a similar manner the convergence of the jacobian of a set of transformations which approach a limiting transformation, or, what is the same thing, of a set of unrestricted functions of a complex variable which approach a limiting function. It is shown that the jacobian of the limiting transformation is the limit of the sequence of jacobians, if the series which arise from partial differentiation of the functions defining the sequence of transformations each converge uniformly. Moreover, it is shown that if the transformations converge uniformly, and if the jacobians converge uniformly, the jacobian of the limiting transformation is the limit approached by the jacobians.

These statements appear in more elegant form with the notation of the theory of functions of a complex variable, in its non-specialized form. Theorems analogous to each of the preceding are stated for an uncountable set of functions which cluster about a given function.

23. In the theory of functions of two complex variables $z = x + iy$, $w = x' + iy'$, the transformations of importance are of the form $Z = \phi(z, w)$, $W = \psi(z, w)$, where ϕ and ψ are general analytic functions. In the four-dimensional space with cartesian coordinates x, y, x', y', these transformations are not conformal, and Poincaré in his recent paper in the *Palermo Rendiconti* therefore employs (provisionally) the term regular. Professor Kasner obtains several geometric characterizations of the regular transformations. Two linear systems of planes are converted into themselves and only the angles situated in these planes are left invariant. The family of surfaces affected conformally is easily derived. The simplest characteristic invariant (pseudo-angle) is connected with the intersection of a line with a three-dimensional variety. In conclusion certain finite subgroups are considered: the 9-parameter conformal group, the linear, and the linear fractional groups.

24. Professor Miller's second paper is composed of two distinct parts. The first is devoted to a list of all the possible modular systems of the form [mod p, $\phi(x)$] by means of which the abstract abelian groups whose orders do not exceed 12 may be represented as congruence groups. This part may be regarded as complementing, along a certain line, the articles of Kempe and Cayley in which all the possible abstract groups whose orders do not exceed 12 are considered at some length. The second part is devoted to a few general theorems relating to the representation of abstract groups in regard to Kronecker's modular systems. Among the theorems proved in this part are the following: It is not possible to represent every abstract abelian group as a congruence group in regard to some modular system. The lowest group which does not have the property that it may be represented by a complete set of prime residues with respect to a modular system is of order 5. If $\phi(x)$ is the

product of k distinct irreducible functions $(\phi_1, \phi_2, \cdots, \phi_k)$, then the residues which are prime mod p, $\phi(x)$ constitute the direct product of the groups with regard to the separate modular systems $p, \phi_1; p, \phi_2; \cdots; p, \phi_k$. The number of the invariants equal to p^α in the group formed by the prime residues mod p, x^n is $\epsilon'[n/(p^{\alpha-1})] - 2\epsilon'[n/(p^\alpha)] + \epsilon'[n/(p^{\alpha+1})]$, where $\epsilon'[x]$ is the largest integer which is less than x. The last theorem is regarded as especially interesting in view of the fact that it exhibits the marked difference between the properties of these groups and those formed by the numbers which are prime to p^m, when they are combined by multiplication and the products are reduced mod p^m.

25. A complex projective space S_n of n (complex) dimensions is defined analytically as the totality of points $(x_0, x_1, \cdots, x_n)$, where the homogeneous coordinates x_i are any complex numbers not all zero. A projective transformation in this space is represented by a linear homogeneous transformation on the x_i with complex coefficients. The totality of points whose coordinates are real form a subspace R_n of n (real) dimensions in S_n. Every subspace of S_n which is obtained by subjecting R_n to a collineation in S_n is called an n-dimensional chain in S_n, or more briefly an n-chain.

In a paper presented to the Society at its last April meeting in New York (abstract in BULLETIN, volume 14 (June, 1908), page 411), Professor Young developed certain properties of the two-chains of a complex plane and applied the results to the classification of the collineations in a plane "with respect to reality." The present paper continues the investigation there begun by effecting a classification "with respect to reality" of the continuous groups of collineations in a plane. The object is to enumerate all types of such groups which leave a two-chain invariant; every such group can evidently be transformed into one in which all the coefficients of every collineation are real. Two groups leaving the same two-chain invariant are regarded as equivalent, if and only if they can be transformed into each other by a collineation leaving the same two-chain invariant. The treatment of the problem is very largely synthetic and on the basis of the results of the first paper and with the help of the known lists of the groups of collineations in a plane (cf. Newson, *American Journal*, volume 24, page 169, and Meyer, Chicago Congress Papers, page 188) offers little difficulty. The author finds seven types of groups in addition to those obviously obtainable from Newson's list by restricting the invariant elements to be real, or from Meyer's list by restricting the coefficients to real values. That the enumeration is complete rests on Lie's theorem that every continuous group with real coefficients gives rise to a complex group of the same number of parameters, if the parameters are allowed to assumed complex values.

26. Mr. Wedderburn shows that, if a group be expressed in two distinct ways as a direct product of prime factors (a group being prime if its only factors are itself and the identity), the factors in one series are simply isomorphic with the factors, taken in a suitable order, in the other series. Further properties of such series are established. The paper will be offered to the *Transactions*.

27. The main result in Professor Fite's paper is that a group all of whose operations, except identity, are of order three is abelian, metabelian, or of class three. It is also shown that such groups of class three exist.

28. In this paper Dr. Moore discusses the summability of the development of an arbitrary function in terms of Bessel functions. Sufficient conditions upon the arbitrary function are obtained that the development should be summable * to the value of the function at every point of the interval $0 < x < 1$ at which the function is continuous and should be uniformly summable throughout any sub-interval of an interval in which the function is everywhere continuous. The convergence factors which occur in connection with these developments in the investigation of problems in the flow of heat are then discussed and the theorems established are used in showing that the formal results obtained by the methods of Fourier really furnish the desired solution of the problems.

29. Professor Townsend called attention to the fact that the conditions usually given for the interchange of the order of differentiation involve the continuity with respect to the two variables taken together of the partial derivatives concerned. He then showed that by a slight modification of the proof given

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* The definition of a summable series here adopted is the same as that used by the writer in a previous paper (Cf. *Trans. Amer. Math. Soc.*, vol. 8 (1907), p. 299).

by Schwarz* it can be shown that whenever $\partial f/\partial x$, $\partial f/\partial y$, $\partial^2 f/\partial x\partial y$ exist and are continuous with respect to each variable separately, the partial derivative $\partial^2 f/\partial y\partial x$ exists also and is equal to $\partial^2 f/\partial x\partial y$.

30. In abstract Professor Newson's paper is as follows: Let T be a linear transformation in n variables, thus:

$$\rho x'_i = a_{1i}x_1 + \cdots + a_{ni}x_n \quad (i = 1, 2, \cdots, n).$$

Let the determinant of T be such that T represents a collineation of the most general type in S_{n-1}, a space of $n - 1$ dimensions. It leaves invariant n linearly independent points A_i, whose coordinates may be represented by $A_{1i}, A_{2i}, \cdots, A_{ni}$ $(i = 1, 2, \cdots, n)$. Let λ_i $(i = 1, 2, \cdots, n)$ be n independent constants such that λ_i/λ_1 $(i = 2, \cdots, n)$ are the cross ratios of the $n - 1$ one-dimensional projective transformations along the invariant lines A_1A_i $(i = 2, \cdots, n)$. The linear transformation T may now be written in the explicit normal form, as follows:

$$T: \qquad \rho x'_i = \begin{vmatrix} x_1 & \cdots & x_n & 0 \\ A_{11} & \cdots & A_{n1} & \lambda_1 A_{i1} \\ \vdots & & \vdots & \vdots \\ A_{1n} & \cdots & A_{nn} & \lambda_n A_{in} \end{vmatrix}.$$

The determinant of the normal form of T is $D = \lambda_1 \cdots \lambda_n \Delta^n$, where $\Delta = |A_{ij}|$. The characteristic equation of the normal form of T breaks up into linear factors, thus:

$$\prod_1^n [\rho + (-1)^n \lambda_i \Delta] = 0.$$

If ρ/Δ be replaced by ρ', the roots of the characteristic equation in ρ' are $(-1)^{n+1}\lambda_i$ $(i = 1, \cdots, n)$.

31. Professor Dowling's paper is in abstract as follows: The equation of any proper sextic having nine double points may be reduced to the form $\lambda S + U^2 = 0$, where $S = 0$ is one such sextic and $U = 0$ is a cubic passing through the nine double points. Elliptic sextics having nine crunodes are of three forms: a single even branch crossing itself nine times, two odd branches, or two even branches.

Choosing λ in such a way that the sextic $\lambda S + U^2 = 0$ possesses the maximum number of connected loops, i. e., loops within which $\lambda S + U^2$ has the same sign (this maximum number is nine), the sextic $\epsilon(\lambda S + U^2) = V^2$, for proper values of ϵ and when U crosses the nine loops of $\lambda S + U^2 = 0$, consists, in general, of nine ovals within the loops of $\lambda S + U^2 = 0$. But when U is unipartite and V is bipartite the sextic $\epsilon(\lambda S + U^2) = V^2$ has in addition two ovals, one within and one without the oval of V.

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* Abhandlungen, vol. II, p. 275.

This method of building up the general sextic from a crunodal elliptic sextic prevents the escape of ovals from the loops of the elliptic sextic and at the same time insures the maximum number of branches which a sextic may have. It moreever shows the necessity for the nested branches of the sextic.

32. Professor Haskins's note points out that the methods used in the proof of the second law of the mean by application of the fundamental theorem of Fourier's constants * apply to the intervals of Lebesgue as well as to those of Riemann.

33. If we write an ordinary differential equation of the first order and nth degree in the minimum coordinates $u = x + iy$, $v = x - iy$, it has the form

$$\pi^n + a_1\pi^{n-1} + a_2\pi^{n-2} + \cdots + a_i\pi^{n-i} + \cdots + a_{n-1}\pi + a_n = 0 \tag{1}$$
$$(\pi = dv/du).$$

A necessary and sufficient condition that this differential equation admit the conformal infinitesimal transformation

$$U'f: \qquad \phi(u)\frac{\partial f}{\partial u} + \psi(u)\frac{\partial f}{\partial v} + \pi[\psi'(v) - \phi'(u)]\frac{\partial f}{\partial \pi}$$

is found from the invariants. There are $n - 1$ invariants involving the coefficients a and these lead to $n - 1$ involving the first partial derivatives of a. For the above property it is then necessary and sufficient that these $2n - 2$ invariants be functions of the same harmonic function.

The theory is applied to the equation of the second degree $\pi^2 + a_1\pi + a_2 = 0$ and to a concrete example of this type.

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Acting Secretary.

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* Cf. Haskins, *Annals of Math.*, ser. 2, vol. 9 (1908), p 173.

ANSWER TO A QUESTION RAISED BY CAYLEY AS REGARDS A PROPERTY OF ABSTRACT GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society, September 10, 1908.)

IN 1859 Cayley * gave an enumeration of the possible abstract groups of order 8, and at the end of the note devoted to this subject he considered briefly the groups defined by two operators s_1, s_2 satisfying the following conditions:

$$s_1^m = 1 \qquad s_2^n = 1 \qquad s_1 s_2 = s_2 s_1^k.$$

He remarks, "the group corresponding to $k = 1$ is distinct from that for any other value of k, but I have not ascertained whether the values other than unity do, or do not, give groups distinct from each other." That different values of k *may* lead to distinct groups is very evident, and we shall assume that the question whose answer Cayley was seeking may be expressed as follows: *Given m and n, and that $k \not\equiv 1 \bmod m$, for what admissible values of k are the groups generated by s_1 and s_2 distinct?* Even if this question should be more general than the one which Cayley had in mind, it relates to such a fundamental matter as to make a direct answer desirable. Partial answers may be found in various places, especially in a comparatively recent paper by Netto, † which is largely devoted to these elementary groups.

The conditions imposed on s_1 and s_2 are equivalent to the conditions that a cyclic group of finite order m is transformed into itself by an operator of finite order n. As a first result we have that the order of G, the group generated by s_1 and s_2, is mn/l, where l is the number of operators common to the two cyclic groups generated by s_1 and s_2 respectively. Cayley implicitly assumed $l = 1$. When $k \equiv 1 \bmod m$, G is either cyclic or the direct product of two cyclic groups. This special case will be excluded in what follows, as it is not included in the question under consideration. As s_1 and s_2 are supposed to be non-commutative, there is some lowest power of s_2, say s_2^r, which is commutative with s_1, r being a divisor of n. Since a cyclic group of order r has $\phi(r)$ generators, $\phi(r)$ being the totient of r, it results that $\phi(r)$ different values of k give rise to the same group whenever one of them may be used for k. That is, all the numbers which may be used for k can be arranged in sets of $\phi(r)$, such that each set corresponds to the same group. It is not difficult to prove that whenever the values of k belong to two such sets the corresponding groups will also be distinct. In other words, there is a (1, 1) correspondence between these sets of $\phi(r)$ numbers and the distinct groups obtained by using for k all the different numbers belonging to exponent r. This theorem gives a complete answer to the question under consideration and its proof is contained in the following two paragraphs.

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* *Philosophical Magazine*, vol. 18 (1859), p. 34.

† *Crelle*, vol. 128 (1905), p. 243.

Let H be the cyclic group generated by s_1. If the values of k corresponding to two distinct sets of $\phi(r)$ numbers gave rise to the same group, the group corresponding to one of these sets would involve at least two invariant subgroups similar to H, each corresponding to a cyclic quotient group and hence involving all the commutators of G. Moreover, these two subgroups H_1, H_2 would have to be transformed differently by the operators of G. Let P be any Sylow subgroup of odd order p^m contained in H_1. If the operators of P are transformed according to a substitution whose order is prime to p, it is necessary that P be common to H_1 and H_2, and hence it is transformed in the same manner under G. If the order of this substitution is not prime to p, the Sylow constituent of order p^α in the commutator subgroup of G is common to H_1 and H_2, and the generators of P and the corresponding subgroup in H_2 are transformed into themselves multiplied by operators of the same order under G.* Hence in every case the Sylow subgroups of odd order in H_1 and H_2 are transformed in the same manner under G.

If P is of order 2^m $(m > 2)$, its group of isomorphisms is the direct product of a cyclic group of order 2^{m-2} and a group of order 2. Hence this group of isomorphisms contains two cyclic subgroups of every order greater than 2 and less than 2^{m-1}, and these two cyclic subgroups correspond to commutator subgroups of different orders. From this it follows that P and the corresponding subgroup in H_2 are transformed in the same manner whenever P is transformed according to a cyclic group whose order exceeds 2. When a generator of P is transformed into its inverse or into its $(2^{n-1}-1)$th power, the commutator subgroup is of order 2^{n-1}. In each of these cases, n must be divisible by 4 to insure more than one invariant P, and each such P must therefore be transformed in the same manner under G. Combining these results we have the theorem:

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* BULLETIN, vol. 7 (1901), p. 350.

If r is the lowest power of s_2 which is commutative with s_1, the numbers which can be used for k are precisely those belonging to exponent r modulo m, and all of these numbers less than m may be arranged into sets of $\phi(r)$ such that those of the same set, but no others, correspond to the same group.

For the sake of seeing more clearly the nature of the problems included under this theorem, we may consider the special case when $m = 72$ and $n = r = 6$. Since the group of isomorphisms of the cyclic group of order 72 contains 14 operators of order 6, there are 14 numbers which can be used for k, viz., 5, 7, 11, 13, 23, 29, 31, 41, 43, 47, 59, 61, 65, 67. As $\phi(6) = 2$, these numbers may be arranged into pairs, each pair leading to the same group. These seven pairs are as follows:

5, 29; 7, 31; 11, 59; 13, 61; 23, 47; 41, 65; 43, 67.

Only three of these pairs lead to groups involving more than one invariant cyclic subgroup of order 72, viz., 7, 31; 13, 61; 43, 67. The first and last of these three groups involve exactly three invariant cyclic subgroups of order 72, while the second involves six such subgroups. In each case, all of these invariant subgroups are transformed in the same manner under G. It should not be inferred that all the invariant cyclic subgroups of order m are transformed in the same manner under G; the theorem merely implies that two invariant cyclic subgroups of order m which are transformed differently under G cannot both correspond to a cyclic quotient group of G.

When n is a prime number p and G contains more than one invariant cyclic subgroup of order m, it is necessary that $r = n$, and in this case it is easy to prove that m is divisible by p^2 when $p > 2$, and by p^3 when $p = 2$. Moreover, all the commutators of G are invariant and there is only one group for given values of m and n which involves more than one invariant cyclic subgroup of order m. In fact, if G contains more than one subgroup of order m, it is necessary that m is divisible by p, since all the operators of G which are not in the subgroup generated by s_1 have orders which are divisible by p. Moreover, two such cyclic subgroups would have in common all their operators whose orders are prime to p. Hence we have that G is the direct product of its Sylow subgroup of order p^m and its other Sylow subgroups whenever it involves more than one subgroup of order m. As the group of isomorphisms of a cyclic group of order p^α, $p > 2$, is cyclic, we have the theorem:

When n is a prime p and G involves more than one subgroup of order m, these subgroups must be cyclic and invariant, and G is the direct product of the non-abelian group of order p^m involving p cyclic subgroups of order p^{m-1}, and a cyclic group whose order is prime to p.

It results from the last theorem that there is one and only one non-abelian group of order pg which contains more than one invariant cyclic subgroup of order g, whenever g is divisible by p^2 or by p^3 as p is odd or even. When g is not divisible by one of these numbers, there is no such group. In particular, there is one and only one non-abelian group of order $2g$ involving two cyclic subgroups of order g whenever g is divisible by 8 and only then. If we let $n = 2$ in the example considered above, where $m = 72$, there are 7 numbers which can be used for k, viz., 17, 19, 35, 37, 53, 55, 71. The seven corresponding groups are distinct and only one of them (when $k = 37$) contains more than one cyclic subgroup of order 72. When $n = r = p^\alpha$, G cannot contain more than one invariant cyclic subgroup of order m unless m is divisible by $p^{\alpha+1}$ or $p^{\alpha+2}$ according as p is odd or even; and all these cyclic subgroups have in common the operators whose orders are prime to p in the subgroup generated by s_1. The number of these invariant subgroups cannot exceed p^β, where β is the largest number not greater than α which satisfies the condition that m is divisible by $p^{\alpha+\beta}$ when $p > 2$, and by $p^{\alpha+\beta+1}$ when $p = 2$.

NOTE ON THE THEOREM OF GENERALIZED FOURIER'S CONSTANTS.

BY PROFESSOR W. D. A. WESTFALL.

In the theory of the development of arbitrary functions $f(x)$ in series of normalized orthogonal functions $\psi_i(x)$,

$$f(x)=\sum_1^\infty a_i\psi_i(x),$$

$$a_i=\int_a^b f(x)\psi_i(x)dx, \qquad \int_a^b \psi_i(x)\psi_k(x)dx\begin{cases}=0, & i\neq k,\\ =1, & i=k,\end{cases}$$

sufficient conditions that this equality exists, and that the series converges uniformly, are in general that $f(x)$ and its first $m-1$ derivatives are continuous in (a, b) and satisfy homogeneous boundary conditions for $x=a$ and $x=b$.* Then there follows immediately the fundamental theorem of "generalized" Fourier's constants

$$(1) \qquad \int_a^b \{f(x)\}^2dx=\sum_1^\infty a_i^2.$$

This note will give a simple proof that, in case (1) holds true for every function satisfying the above conditions, it holds true for every integrable function $f(x)$, such that $\{f(x)\}^2$ is integrable.†

Since $f(x)$ and $\{f(x)\}^2$ are integrable in (a, b) there exists, for every $\epsilon>0$, a division of (a, b) in a finite number of subintervals $(x_1, x_2), (x_2, x_3), \cdots, (x_{n-1}, x_n)$, $x_1=a$, $x_n=b$, such that a function $\phi(x)$ can be defined, having the following properties:

$$\begin{gathered}|\phi(x)| = \text{lower bound of } f(x) \text{ in } (x_i, x_{i+1}) \text{ for } x_i \leqq x < x_{i+1},\\ (2) \qquad \phi(x)f(x)\geqq 0,\\ \left|\int_a^b [\{f(x)\}^2-\{\phi(x)\}^2]dx\right|<\frac{\epsilon}{2}.\end{gathered}$$

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* D. Hilbert, "Zweite Mitteilung über Integralgleichungen," *Göttinger Nachrichten*, 1904.
E. Schmidt, Dissertation, Göttingen, 1905.

† The theorem has been proven essentially by W. Stekloff with the restriction that $f(x)$ be bounded, *Mémoires de l'Académie de St. Pétersbourg*, 1904. The above proof is simpler and does away with this restriction.

Consider now a function $F(x)$ defined as follows: in the δ-neighborhood of x_i,

$$F(x)=\phi(x_i+\delta)\frac{\int_{x_i}^{x}(x-x_i)^m(x-\delta-x_i)^m dx}{\int_{x_i}^{x_i+\delta}(x-x_i)^m(x-\delta-x_i)^m dx},\quad (x_i\leqq x\leqq x_i+\delta),$$

$$F(x)=\phi(x_i-\delta)\frac{\int_{x_i}^{x}(x-x_i)^m(x+\delta-x_i)^m dx}{\int_{x_i}^{x_i-\delta}(x-x_i)^m(x+\delta-x_i)^m dx},\quad (x_i-\delta\leqq x\leqq x_i),$$

where $0<\delta<\text{minimum }\frac{1}{2}(x_k-x_{k-1})$, $(i, k=1, 2, \cdots, n)$. For other values of x, $F(x)=\phi(x)$. $F(x)$ and its first $m-1$ derivatives are continuous and vanish for $x=a=x_1$ and $x=b=x_n$. Moreover,

$$(3)\qquad |F(x)|\leqq|\phi(x)|,\quad F(x)\phi(x)\geqq 0.$$

Hence

$$\left|\int_a^b\{\phi(x)\}^2-\{F(x)\}^2dx\right|\leqq\int_{x_1}^{x_1+\delta}\phi^2(x)dx+\int_{x_n-\delta}^{x_n}\phi^2(x)dx$$
$$+\sum_2^{n-1}\int_{x_\nu-\delta}^{x_\nu+\delta}\phi^2(x)dx.$$

Since $\phi(x)$ is bounded, δ can be chosen so small that

$$(4)\qquad \int_a^b\{\phi(x)\}^2-\{F(x)\}^2dx\leqq\tfrac{1}{2}\epsilon.$$

From (2), (3), (4)

$$|F(x)|\leqq|f(x)|,\quad F(x)f(x)\geqq 0,$$

$$(5)\qquad \left|\int_a^b\{f(x)\}^2-\{F(x)\}^2dx\right|<\epsilon,$$

$$\{f(x)-F(x)\}^2\leqq\{f(x)\}^2-\{F(x)\}^2.$$

From a known theorem there exists the following inequality for any set of continuous normalized orthogonal functions $\psi_i(x)$ and any integrable function $f(x)$ such that $|f(x)|$ is also integrable:

$$(6) \qquad \int_a^b \{f(x)\}^2 dx \geqq \sum_1^\infty a_i^2, \quad a_i = \int_a^b f(x)\,\psi_i(x)dx.$$

Hence the identity exists

$$(7) \quad \int_a^b \{f(x)\}^2 dx - \sum_1^\infty a_i^2 = \int_a^b [\{f(x)\}^2 - \{F(x)\}^2]\,dx$$

$$+ \int_a^b \{F(x)\}^2 dx - \sum_1^\infty A_i^2 + \sum_1^\infty (A_i - a_i)(A_i + a_i).$$

$$a_i = \int_a^b f(x)\,\psi_i(x)dx, \quad A_i = \int_a^b F(x)\,\psi_i(x)dx.$$

Moreover

$$\left|\sum_1^\infty (A_i - a_i)(A_i + a_i)\right| \leqq \sqrt{\sum_1^\infty (A_i - a_i)^2 \cdot \sum_1^\infty (A_i + a_i)^2}$$

or from (6)

$$\leqq \sqrt{\int_a^b \{F(x) - f(x)\}^2 dx \cdot \int_a^b \{F(x) + f(x)\}^2 dx}$$

$$< \sqrt{\epsilon \int_a^b \{2f(x)\}^2 dx} \leqq 2\sqrt{\epsilon \int_a^b \{f(x)\}^2 dx},$$

since $|F(x)| \leqq |f(x)|$ and $F(x)f(x) \geqq 0$.

Applying this inequality with (5) in (7),

$$(8) \quad \left|\int_a^b \{f(x)\}^2 dx - \sum_1^\infty a_i^2\right| \leqq \epsilon + \int_a^b \{F(x)\}^2 dx - \sum_1^\infty A_i^2 dx$$

$$+ 2\sqrt{\epsilon \int_a^b \{f(x)\}^2 dx}.$$

This inequality holds for any $\epsilon > 0$ and the corresponding function $F(x)$. Hence the theorem:

If for a set of normalized orthogonal functions $\psi(x)$ and every function $f(x)$, which with its first $m - 1$ derivatives is continuous in (a, b) and satisfies for $x = a$, $x = b$ a set of homogeneous boundary conditions, there exists the equality

$$\int_a^b \{f(x)\}^2 dx = \sum_1^\infty \left\{\int_a^b f(x)\psi_i(x)dx\right\}^2,$$

then this equality holds true for every integrable function $f(x)$ such that $\{f(x)\}^2$ is integrable.

GÖTTINGEN,
July 17, 1908.

ON THE LOGICAL BASIS OF GRASSMANN'S EXTENSIVE ALGEBRA.

BY MR. A. R. SCHWEITZER.

§ 1.

IN studying the algebra of Grassmann fundamentally, we must carefully distinguish between the Ausdehnungslehre proper and the Ausdehnungslehre in a broad sense. Grassmann himself makes no rigid separation of the two viewpoints; generally, however, the former is found in the edition of 1844 and the latter is in the edition of 1862 and in various memoirs.* Briefly, we may say that the Ausdehnungslehre proper for n dimensions ($n = 1, 2, 3, \cdots$) is a development of n-dimensional euclidean geometry by means of the outer product of $n + 1$ points, which fundamentally is reducible to sameness of sense of two $(n + 1)$-hedra. It consists of descriptive axioms and certain axioms which relate exclusively to n-spatial congruence. On the basis of these axioms and their consequences, we arrive at the broader conception of the Ausdehnungslehre by means of suitable abstraction, the introduction of parameters, "formalization," etc.†

§ 2.

If we take three dimensions, the fundamental properties of the Ausdehnungslehre are as follows. Concretely expressed, the basal relation is sameness of sense of two tetrahedra (identical or not) which is implied by what Grassmann has called "Gleichbezeichnung." ‡ This relation is fundamentally nonmetrical, and solely in terms of it we may construct a system of postulates for three-dimensional descriptive geometry which is sufficient for projective geometry. That two tetrahedra are "gleichartig"* implies that they have the same or opposite senses, *i. e.*, they are cospatial. Also four points α, β, γ, δ form a tetrahedron if and only if the tetrad $\alpha\beta\gamma\delta$ is in the relation of sameness of sense to itself; abstractly this is expressed by $\alpha\beta\gamma\delta \mathrm{K} \alpha\beta\gamma\delta$ or $\alpha\beta\gamma\delta\mathrm{K}$. We note that the element point and relation K are undefined symbols which are effective under the conditions specified in the postulates which involve them. If then $\alpha\beta\gamma\delta\mathrm{K}$, we have $\beta\alpha\gamma\delta\mathrm{K}$, $\alpha\beta\gamma\delta\mathrm{K}\beta\gamma\alpha\delta$, $\alpha\beta\gamma\delta\mathrm{K}\gamma\delta\alpha\beta$, but $\alpha\beta\gamma\delta\overline{\mathrm{K}}\beta\alpha\gamma\delta$; † *i. e.*, $\alpha\beta\gamma\delta$ and $\beta\gamma\alpha\delta$ have the same sense, etc. The axiom of dimensionality is that $\alpha\beta\gamma\delta\mathrm{K}$, $\alpha'\beta'\gamma'\delta'\mathrm{K}$ imply $\alpha\beta\gamma\delta\mathrm{K}\alpha'\beta'\gamma'\delta'$ or $\alpha\beta\gamma\delta\mathrm{K}\beta'\alpha'\gamma'\delta'$. This axiom is a characteristic of sameness of sense as distinguished from similarity of direction (gleichläufig).‡ Axioms of geometric connection are: if $\alpha\beta\gamma\delta$ is a tetrahedron, and ξ is any point, then $\xi\beta\gamma\delta$ or $\alpha\xi\gamma\delta$ or $\alpha\beta\xi\delta$ or $\alpha\beta\gamma\xi$ is a tetrahedron; if $\alpha\beta\gamma\delta$ and $\xi\beta\gamma\delta$ are two tetrahedra and $\xi \neq \alpha$, then $\alpha\xi\gamma\delta$ or $\alpha\beta\xi\delta$ or $\alpha\beta\gamma\xi$ is a tetrahedron; if $\alpha\beta\gamma\delta$ is a tetrahedron and ξ lies on the faces $\alpha\beta\gamma$ and $\alpha\beta\delta$, then ξ is collinear with $\alpha\beta$ (*i. e.*, $\alpha\beta\xi\delta\overline{\mathrm{K}}$ for any δ). The following existential axioms are needed: there exists the tetrahedron $\alpha_0\beta_0\gamma_0\delta_0$; if $\alpha\beta\gamma\delta$ is a tetrahedron, there exists a point ξ such that α is in the interior of the tetrahedron $\beta\gamma\delta\xi$; if ξ is in the interior of $\alpha\beta\gamma\delta$, there exists an η such that η is on the plane $\xi\gamma\delta$ and between $\alpha\beta$.‖ These axioms have a broader logical significance: the relation of sameness of sense is linearly transitive and symmetrical; also the tetrahedra $\xi\beta\gamma\delta$ and $\alpha\beta\gamma\delta$ have the same sense if ξ is collinear with $\alpha\beta$ and $\xi\beta\gamma\delta$ and $\alpha\xi\gamma\delta$ have the same sense, or if ξ is on the plane $\alpha\beta\gamma$ and $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$ have the same sense, or if the tetrahedra $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ have the same sense. Finally, if we add to the above properties the property of Dedekind continuity, which can be easily formulated in terms of sameness of sense in several elegant ways, we have all the descriptive properties necessary.§

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* Cf. the Collected Works of Grassmann. For references to the Ausdehnungslehre and related subjects we may refer to Macfarlane's admirable bibliography, Dublin, 1904.

† For instance, see *Crelle*, vol. 49, p. 123; *Math. Annalen*, vol. 12, p. 376; Ausdehnungslehre, 1862, §§ 151–215; Study, *Wiener Berichte*, vol. 91, p. 111.

‡ Coll. Works, I, p. 303, 304.

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* Grassmann, Coll. Works, I, p. 163.

† The rule over the K is a symbol of negation.

‡ Grassmann, Coll. Works, I, p. 49, note 3.

‖ For the definition of betweenness, etc., cf. the definitions for the plane by the author, BULLETIN, November, 1906.

§ For the precise abstract statement of these properties, proof of their sufficiency, as well as full formulation of the definitions of betweenness, complanarity, etc., see a paper by the author offered to the *Transactions* for publication.

§ 3.

To the terms of the preceding relation of sameness of sense we give the character of magnitudes by expressing $\alpha\beta\gamma\delta\mathrm{K}\alpha'\beta'\gamma'\delta'$ by $\alpha\beta\gamma\delta \equiv \alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' \cdot \alpha'\beta'\gamma'\delta'$ and postulating that the symbols $(\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta')$ are a complete set of real positive numbers (k).[*] The statement $\alpha\beta\gamma\delta\overline{\mathrm{K}}$ is expressed by $\alpha\beta\gamma\delta \equiv 0 \cdot \alpha'\beta\gamma'\delta'$, *i. e.*, $\alpha\beta\gamma\delta \equiv 0$; $\alpha\beta\gamma\delta\mathrm{K}$ is expressed by $\alpha\beta\gamma\delta \equiv \mathrm{I} \cdot \alpha\beta\gamma\delta$ or $\alpha\beta\gamma\delta \not\equiv 0$. We can now verify certain portions of §§ 96, 97 of Grassmann's Ausdehnungslehre of 1844; for example, if $\alpha\beta\gamma\delta \not\equiv 0$, then $\alpha\beta\gamma\delta \equiv \mathrm{k}\ \alpha_0\beta_0\gamma_0\delta_0$ where k is a real number;[†] if $\alpha_1\beta_1\gamma_1\delta_1 \equiv \mathrm{k}_1\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha_2\beta_2\gamma_2\delta_2 \equiv \mathrm{k}_2\alpha_0\beta_0\gamma_0\delta_0$ then for the sum we have $\alpha_1\beta_1\gamma_1\delta_1 + \alpha_2\beta_2\gamma_2\delta_2 \equiv (\mathrm{k}_1 + \mathrm{k}_2)\,\alpha_0\beta_0\gamma_0\delta_0$, etc. Other properties of spatial congruence are that if $\alpha\beta\gamma\delta \not\equiv 0$, $\alpha\beta\gamma\delta' \not\equiv 0$ and $\delta\delta'$ is parallel to $\alpha\beta\gamma$, then $\alpha\beta\gamma\delta \equiv \mathrm{I} \cdot \alpha\beta\gamma\delta'$; and that the sum $\xi\beta\gamma\delta + \alpha\xi\gamma\delta + \alpha\beta\xi\delta + \alpha\beta\gamma\xi \equiv \mathrm{I} \cdot \alpha\beta\gamma\delta$. We are now able to prove the grassmannian numerical derivation of points, uniqueness of parallel line, derivation of vectors, etc.[‡] The further procedure consists in admitting the point with weight zero and developing projective geometry. Grassmann clearly indicates the character of this development in the Ausdehnungslehre of 1844. In this connection one is reminded of a hiatus in Grassmann's calculus, namely, the absence of an absolute theory of angles. Such a theory is certainly desirable from a pedagogical as well as theoretical viewpoint; and the prospect for it on the basis of the preceding properties is very promising. However, from the above, we are already in a position to verify in a fundamental, logical manner that "Grassmann's Ausdehnungslehre is a shape into which projective geometry may be thrown."[§]

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[*] Cf. Grassmann, Coll. Works, I, p. 138.

[†] If $\mathrm{k} > 0$ then $\alpha\beta\gamma\delta$ and $\alpha_0\beta_0\gamma_0\delta_0$ are "gleichbezeichnet"; if $\mathrm{k} \neq 0$, $\alpha\beta\gamma\delta$ and $\alpha_0\beta_0\gamma_0\delta_0$ are "gleichartig."

[‡] Cf. Grassmann, Coll. Works, I, § 110; Peano, *Calcolo Geometrico.*

[§] Lasker, *Proc. London Math. Society*, vol. 28, pp. 217, 218.

GENERAL ALGEBRAIC SOLUTIONS IN THE LOGIC OF CLASSES.

BY PROFESSOR L. M. HOSKINS.

(Read before the San Francisco Section of the American Mathematical Society, February 29, 1908.)

THE following treatment of the problem of inference in the logic of classes possesses some interest from its analogy to general solutions in ordinary algebra. The character of the general solutions here considered is most simply illustrated by what may be called the generalized problem of the syllogism, which may be stated as follows:

Let x, y, z be three class symbols, and let

$$f_1(x, y) = 0, \quad f_2(y, z) = 0,$$

be any two propositions involving x, y and y, z respectively; then it is required to deduce a proposition

$$f_3(x, z) = 0$$

involving x and z but not y.

The most general forms of the above propositions are (writing x' for $1 - x$, etc.)

$$(1) \qquad f_1(x, y) = l_1xy + l_2xy' + l_3x'y + l_4x'y' = 0,$$

$$(2) \qquad f_2(y, z) = m_1yz + m_2yz' + m_3y'z + m_4y'z' = 0,$$

$$(3) \qquad f_3(x, z) = n_1xz + n_2xz' + n_3x'z + n_4x'z' = 0,$$

in which l, m, n are numerical coefficients; and the non-vanishing of any coefficient (as m_2) implies the vanishing of the corresponding class term (yz'). The problem is to express the coefficients in (3) in terms of those in (1) and (2).

A solution is obtained in simple and symmetrical form by regarding (1), (2), and (3) as particular cases of the most general proposition involving x, y, z,

$$(4) \quad f(x, y, z) = axyz + bxyz' + cxy'z + dxy'z' + ex'yz + fx'yz' + gx'y'z + hx'y'z' = 0.$$

By Boole's rule of elimination

$$f_1(x, y) = f(x, y, 1)f(x, y, 0).$$

Hence (1), (2), and (3) may be written

$$(5)\qquad abxy + cdxy' + efx'y + ghx'y' = 0,$$

$$(6)\qquad aeyz + bfyz' + cgy'z + dhy'z' = 0,$$

$$(7)\qquad acxz + bdxz' + egx'z + fhx'z' = 0.$$

It is now easy by inspection to determine whether the non-vanishing of particular coefficients in (5) and (6) implies the non-vanishing of any coefficients in (7). For example, the non-vanishing of ab and cg implies the non-vanishing of ac; *i. e.*, the premises

$$xy = 0, \quad y'z = 0$$

imply the conclusion $xz = 0$. On the other hand, no conclusion can be drawn from

$$xy = 0, \quad yz' = 0$$

since the non-vanishing of ab and bf does not require the non-vanishing of any coefficient in (7).

The above refers primarily to so-called universal propositions; but the solution includes also particular propositions, if these are understood as affirming the existence of the classes referred to. Thus in (5), if any coefficient, as cd, is made zero *while all others remain arbitrary*, the proposition affirms the existence of the class xy', since the four classes xy, xy', $x'y$, $x'y'$ cannot all be assumed to vanish without denying the existence of the "universe of discourse." As an example of inference when one premise is particular, notice that if cd vanishes, while cg does not, we must have $d = 0$ and therefore $bd = 0$; *i. e.*, from the premises

$$xy' \neq 0, \quad y'z = 0,$$

may be inferred the conclusion $xz' \neq 0$.

Propositions involving more than three primary class terms may be treated in like manner, but the application to particular cases becomes rapidly less simple as the number of primary classes increases.

The method applies also to the case in which x, y, z represent propositions instead of classes; but interpretations in the logic of propositions involve some peculiar subtleties.

A GENERAL DIAGRAMMATIC METHOD OF REPRESENTING PROPOSITIONS AND INFERENCE IN THE LOGIC OF CLASSES.

BY PROFESSOR L. M. HOSKINS.

(Read before the San Francisco Section of the American Mathematical Society, February 29, 1908.)

THE following method of representing class relations depends upon an analogy which is most simply explained in the case of propositions involving three class terms x, y, z.

Let the domain of any class x and that of its negative x' be represented by two parallel faces of a cube, and let the other pairs of faces in like manner represent the domains of y, y' and z, z' respectively. Then the domain common to any two of the six classes x, x', y, y', z, z' may be represented by the edge in which the corresponding planes intersect, while the domain

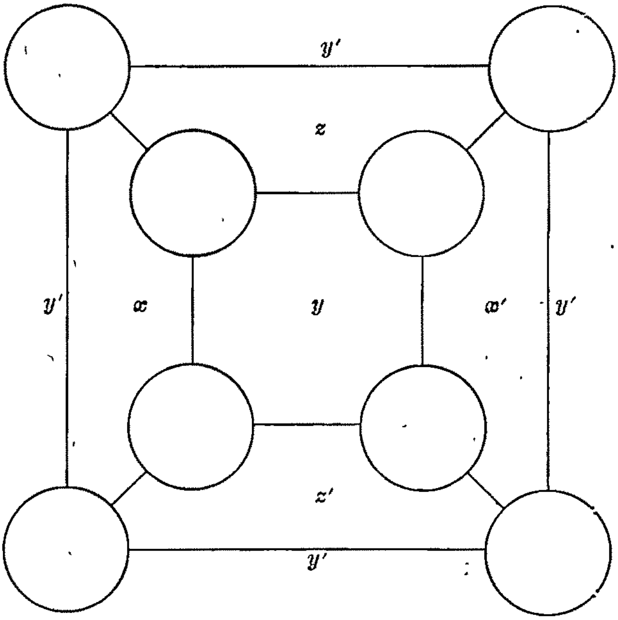

FIG. 1.

common to three of them will be represented by the vertex in which three planes intersect. This gives a symmetrical representation of the relations among the eight subclasses xyz, xyz', $\cdots$.

For practical use in the solution of logical problems, a convenient diagram is obtained by drawing a plane projection of a cube and marking each face with its corresponding class symbol. At each vertex may be left a circular space which, in any given case, may receive a distinguishing mark in accordance with the import of a particular proposition. Such a diagram is represented in Fig. 1.

A symmetrical representation for the case of n primary class terms would require a figure in n dimensions. But a plane diagram for practical use may be made by repeating the diagram for the case of three terms. For example, the case of six primary terms may be represented by eight cubes, each placed at a vertex of a larger cube. Thus the projection shown in Fig. 1 might be drawn on a large scale, and a similar projection drawn in each of the circular spaces; each of these smaller figures representing the relations of three primary terms u, v, w and their negatives.

For practical use a permanent diagram may be drawn upon a slate or blackboard, so that the marks used in particular problems may readily be erased.

HEINRICH MASCHKE: HIS LIFE AND WORK.

Heinrich Maschke was born at Breslau in Germany on October 24, 1853. He received his early education at the Maria-Magdalenen-Gymnasium of his native town, where his exceptional mathematical talent soon attracted the attention of his teachers. In 1872 he graduated from the gymnasium and entered the University of Heidelberg as a student of mathematics, the science which he had selected for his life study. After several semesters at Heidelberg, where he was initiated by Königsberger into the mysteries of infinitesimal calculus, and at Breslau, where he served his year as "Einjährig-Freiwilliger" in the Prussian army, he went for three years to to Berlin, attracted by the famous triad Weierstrass, Kummer, and Kronecker, of whom Kummer seems to have had the most lasting influence upon his mathematical development.

In 1878 he passed with high distinction the government examination (Staatsexamen) in Berlin and in 1880 the doctor's examination in Göttingen with a thesis "Ueber ein dreifach orthogonales Flächensystem, gebildet aus Flächen dritter Ordnung."

Soon afterwards he accepted a position as teacher of mathematics in the Luisenstädtische Gymnasium in Berlin. But though he was a most successful teacher, he soon felt that the work in the gymnasium, which consisted in teaching, for about twenty hours a week, arithmetic and the first elements of algebra and geometry, could not permanently satisfy him. As time went on, this feeling increased, especially after his return to the gymnasium, in 1887, from a year's leave of absence spent at the University of Göttingen under Professor Felix Klein. But there was no hope that he could ever cross the almost insurmountable barriers between the gymnasium and the university, and only a radical step could bring the deliverance from an occupation which had more and more become an irksome burden.

Shortly before, two of his nearest friends had, under similar circumstances, emigrated to the United States and soon found suitable positions. Their success and urgent persuasion determined Maschke to throw up his position and follow their example. But as he did not wish to rely entirely upon the doubtful chances of finding a university position, he employed his leisure hours during the last year of his stay at the gymnasium in studying electrotechnics at the Polytechnicum in Charlottenburg. In July, 1890, he left the gymnasium for good, spent several months in practical electrical work in the Berliner Allgemeine Electricitätsgesellschaft, and completed his electrotechnical studies during the winter semester 1890/91 at the Polytechnicum in Darmstadt under Professor Kittler.

Thus prepared he landed in New York April 1, 1891. Soon afterwards he found a position as electrician in the Weston Electric Instrument Company of Newark, N. J., and one year later he accepted a call as assistant professor of mathematics in the newly founded University of Chicago.

Here at last he found the place best suited to his abilities, where he could freely develop his powers in a sphere of far-reaching usefulness,* and he has always gratefully acknowledged his deep indebtedness to this hospitable country of great opportunities which has made it possible for him to live out his own life.

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* More than seventy of the present members of the American Mathematical Society have, at one time or another, attended Maschke's courses.

With love and enthusiasm, and with the greatest success, he devoted himself to his new duties. His rare pedagogical talent, aided by his long experience in the gymnasium, his genuine scholarly spirit, a strong artistic sense which gave a special charm to his lectures, together with his fine personal qualities combined to make him an ideal teacher. His courses covered a wide range of different subjects: synthetic and analytic geometry, algebraic curves and differential geometry; function theory, elliptic functions, linear differential equations; theory of equations, invariants, groups; analytical mechanics; potential, electricity, — are only a partial list of the subjects on which he lectured in the course of the nearly sixteen years which he was connected with the University of Chicago, first as assistant professor, then as associate, and finally as full professor.

As an investigator Maschke has made his mark in two different branches of mathematics: in the theory of finite groups of linear substitutions and in the theory of quadratic differential quantics. A detailed appreciation of his contributions will be given below; they are all distinguished by a peculiar beauty and elegance, emanating from that same artistic sense which made his lectures so attractive.

Through his teaching, his publications, and his activity in the American Mathematical Society, especially the Chicago Section, Maschke has had a considerable share in the modern movement in mathematics in this country. He was a member of the Council in 1902–1905, and Vice-President of the Society during 1907. In the same connection his address "On present problems of algebra and analysis" at the St. Louis international congress of 1904 should be mentioned.

Many of the readers of these lines have known him personally; they all remember his sunny disposition, his genuine kindness and modesty, qualities which quickly won him the hearts of all with whom he came in contact. But the peculiar charm of his personality lay in its harmoniousness as a whole, in the beautiful balance of his mental and moral faculties which was the source of his quiet strength and manly dignity.

Maschke had always enjoyed excellent health, and there was every prospect that he would live to old age. But it was not to be. Towards the end of 1907 symptoms of an internal disorder began to show themselves; an examination which took place in the last week of February, 1903, revealed the necessity of an immediate operation, to which he succumbed on March 1.

Maschke's career as an investigator began in 1886 at Göttingen under the inspiring influence of Professor Felix Klein. Only two years before, Klein's Vorlesungen über das Ikosaeder had been published, and in the meantime Klein * had developed the extension of his solution of the general equation of the fifth degree, in which the solution of an algebraic equation of a given group is considered as a special case of the so-called form problem for a finite group of linear substitutions, *i. e.*, the problem of determining the values of the independent variables of the substitution group when the values of the invariant forms of the group are given. Hence the great interest which attached to the problem: to determine the complete system of invariant forms for a given finite group of linear substitutions. For the binary groups this problem had been solved by Schwarz, Klein, and Gordan, for the ternary G_{168} by Klein; it was still unsolved for the remaining ternary groups and for the quaternary groups then known. The most interesting of the latter had been derived by Klein, partly from line geometry, partly from the transformation theory of hyperelliptic functions, viz., 1) the so-called group of the Borchardt moduli, a group of $64 \cdot 720$ substitutions, isomorphic with the alternating group of six letters; 2) the group of Klein's hyperelliptic moduli $Z_{\alpha\beta}$ for the transformation of the third degree, a group of 51840 substitutions, isomorphic with the group of the equation of the 27 straight lines on the general cubic surface; 3) a group of $2 \cdot 7!/2$ substitutions, derived by Klein from the consideration of "supernumerary" line coordinates and isomorphic with the alternating group of seven letters.

The first problem which Maschke took up, while still at Göttingen, was the determination of the *complete system of invariant forms for the group of the Borchardt moduli.*† Considering the high order of the group, the problem seems at first sight one of appalling difficulty. But it was precisely this character of the problem which attracted Maschke; he was a very adroit and indefatigable calculator and he delighted in attacking a problem of apparently insurmountable difficulty, reducing first by preliminary ingenious devices the amount of necessary computation to a minimum and taking finally the rest by sheer force. After having derived a system of generating substitutions for his group G in a particularly simple and handy form, he first determines without much difficulty the invariant forms for a certain invariant subgroup of G of order 64. They are five in number and of degree four, and they are linearly substituted among themselves under the application of the main group G. Next he introduces a linear combination Φ of these five forms which is six-valued under a subgroup of G of index 2, and out of these six values of Φ the invariant forms of G are finally built up.

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* Compare Wiman's article in the Encyklopädie, I B 3 f.

† "Ueber die quaternäre endliche, lineare Substitutionsgruppe der Borchardtschen Moduln," *Göttinger Nachrichten*, 1887, p. 421, and *Math. Annalen*, vol. 30, p. 496 (June, 1887).

Maschke made an interesting application of his results to the *solution of the general equation of the sixth degree.** By means of a Tschirnhausen transformation he identifies the general equation of the sixth degree with the resolvent equation satisfied by the six values of the function Φ mentioned above, and by combining this result with the known expressions for the invariants of a sextic in terms of the corresponding hyperelliptic ϑ-zero values, he obtains a solution of the general equation of the sixth degree in terms of hyperelliptic ϑ-zero values, analogous to Hermite's solution of the general quintic equation in terms of elliptic modular functions.

The determination of *the complete system of invariant forms of the quaternary group of 51840 substitutions*,† which Maschke attacked next after his return to the gymnasium, was still more complex; but he succeeded also in this case by similar methods. Here he starts by determining the complete system of invariant forms for that subgroup which leaves the plane $z_0 = 0$ unchanged and which turns out to be the hessian group of 6.216 ternary substitutions. He thus solves incidentally an important problem of the theory of ternary groups.‡

Also for the quaternary group of $2 \cdot 7!/2$ substitutions

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* "La risoluzione della equazione di sesto grado," *Rend. della R. Acc. dei Lincei*, vol. 4, p. 181 (March, 1886). Another solution of the general sextic equation has been derived from Maschke's resolvent for the function Φ by Brioschi; see the references in the Encyklopädie, I B 3 f, p. 550.

† "Aufstellung des vollen Formensystems einer quaternären Gruppe von 51840 linearen Substitutionen," *Göttinger Nachrichten*, 1888, p. 78, and *Math. Annalen*, vol. 33, p. 317 (June, 1888).

‡ For still another ternary group or rather class of groups Maschke has determined the complete system of invariant forms, viz., for those so-called "monomial groups" which are generated by two substitutions of the form

Maschke attempted to determine the complete system of invariant forms. He succeeded * in solving, as a preliminary step, the problem for *the subgroup of 2.168 quaternary substitutions* isomorphic with Klein's ternary G_{168}, but the solution of the problem for the main group seems to have baffled his efforts. Still his occupation with this group led to an interesting geometrical investigation, the study of *a configuration of 140 straight lines* † defined as follows: Let $x_0, x_1, \ldots, x_6$ be supernumerary line coordinates connected by the two relations

$$\sum_{i=0}^{6} x_i = 0\,, \quad \sum_{i=0}^{6} x_i^2 = 0\,;$$

then according to a general theorem of Klein's, every even permutation between the x's is equivalent to a collineation between the point coordinates, and the totality of these collineations constitute the group in question, viewed as a collineation group.

If these operations are applied to a line for which $x_1 = x_2 = x_3$, $x_4 = x_5 = x_6$, the line takes only 140 different positions, and this is the configuration which Maschke studies in detail. Among his results I will only mention that every line L of the configuration is met by 36 other lines of the configuration; these 36 transversals intersect each other in hexads in 6 points of the line L having a peculiar position which Maschke calls *metharmonic*; ‡ they not only lie in three ways in involution, forming in proper notation the pairs

(1, 4) (2, 6) (3, 5); (1, 5) (2, 4) (3, 6); (1, 6) (2, 5) (3, 4), but at the same time (1, 5) is harmonically divided by (2, 4); (2, 6) by (3, 5); (3, 4) by (1, 6).

After the completion of the paper just mentioned, Maschke's scientific productivity suffered an interruption of several years

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$$z_1' = z_2, \quad z_2' = z_3, \quad z_3' = z_1,$$
$$z_1' = a_1 z_1, \quad z_2' = a_2 z_2, \quad z_3' = a_3 z_3,$$

see "On ternary substitution-groups of finite order which leave a triangle unchanged," *Amer. Journal of Math.* vol. 17, p. 168 (December, 1894).

* In 1888, published several years later in the Chicago Congress papers (1893), p. 175 under the title "The invariants of a group of 2.168 linear quaternary substitutions."

† "Ueber eine merkwürdige Configuration gerader Linien im Raum," *Göttinger Nachrichten*, 1889, p. 384, and *Math. Annalen*, vol. 36, p. 190 (July, 1889).

‡ Later on Maschke returned to such systems of metharmonic points in the paper "On systems of six points lying in three ways in involution" (*Annals of Mathematics*, vol. 10, p. 22, December, 1895), where he studies the properties of these point systems interpreted in the complex plane.

through his branching off into electrotechnics. The next paper* of importance is dated November, 1895; it is an *application of Cayley's representation of finite groups by color diagrams to the rotation groups of the regular bodies in three- and four-dimensional space.* The results are remarkably simple and elegant; for instance, the color diagrams in space for the groups of the three-dimensional regular polyhedra are simply obtained by truncating the vertices, choosing the sides of the polygons thus introduced for black lines properly provided with arrows, and the remaining portions of the sides of the original polyhedron for red lines without arrows. The space diagrams thus constructed can be so projected into a plane that in the projection no two vectors intersect, a circumstance which makes these plane diagrams very perspicuous and useful for the study of the groups in question.

During the following years Maschke returned to the theory of finite groups of linear substitutions, but his papers have an essentially different character from his earlier investigations in the same field. He turns from the consideration of special groups to the discovery of general theorems on finite groups of linear substitutions, some of which have proved of the greatest importance in the further development of the theory.

The first of these theorems† states that *every finite group of linear substitutions can be so transformed that in all its substitutions all the coefficients are cyclotomic, i. e.*, rationally expressible in terms of roots of unity, provided that the group contains at least one substitution whose multipliers are all different. The starting point for his proof is the theorem due to Jordan and Frobenius that for every linear substitution of finite period the multipliers are roots of unity, and he establishes his theorem step by step, first for the sum of the diagonal terms, then for the diagonal terms themselves, next for the terms of the first column, and finally for all the coefficients.

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* "The representation of finite groups, especially of the rotation groups of the regular bodies of three- and four-dimensional space, by Cayley's color diagrams," *Amer. Journal of Math.*, vol. 18, p. 156, and *Göttinger Nachrichten*, 1896, p. 1 (November, 1895).

† "Ueber den arithmetischen Charakter der Coefficienten der Substitutionen endlicher linearer Substitutionsgruppen," *Math. Annalen*, vol. 50, p. 492 (November, 1897). The paper is a sequel to the note "Die Reduction homogener Substitutionen von endlicher Periode auf ihre kanonische Form," *Math. Annalen*, vol. 50, p. 220 (March, 1897). Maschke's cyclotomic theorem has since been generalized in various directions by Burnside and Schur.

Perhaps of even greater importance is the following theorem * to which Maschke was led in the course of the proof of his cyclotomic theorem: *Every finite group of linear substitutions, all of whose substitutions contain in the same place (not in the principal diagonal) a coefficient equal to zero, is intransitive, i. e.*, it can be so transformed that the new variables fall into a number of sets such that the variables of each set are transformed among themselves. In Burnside's terminology, the essential part of the theorem may be briefly formulated as follows: *Every group of linear substitutions of finite order is completely reducible.*

In his proof Maschke makes use of the theorem which had been discovered shortly before by Loewy and Moore, that every finite group of linear substitutions leaves at least one definite hermitian form unchanged.

In the same year Maschke completed still another investigation on linear substitutions, viz., *the determination of all ternary and quaternary collineation groups which are holoedrically isomorphic with the symmetric and alternating permutation groups.*† His solution of this beautiful problem is based upon the theorem on hermitian forms just mentioned and upon a theorem due to Moore, giving the generational relations for the abstract groups holoedrically isomorphic with the symmetric and alternating groups. By a skilful application of these two theorems Maschke obtains with comparatively little computation the complete solution of the problem.

During the winter of 1900, while he was giving a course on differential geometry, Maschke discovered a *symbolic method for the treatment of differential quantics*, and his scientific activity during the remaining years of his life was devoted to a detailed and systematic development of this discovery.

Maschke starts from the remark that if $F^1, F^2, \cdots, F^n$ are n invariants of the quadratic differential quantic

$$A = \sum a_{ik} dx_i dx_k, \quad (a_{ki} = a_{ik}, \; i, k = 1, 2, \cdots, n),$$

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* See first reference in the last foot-note, where the theorem is proved under the same restricting assumption which is made in the cyclotomic theorem, and "Beweis des Satzes, dass diejenigen endlichen linearen Substitutionsgruppen, in welchen einige durchgehende Nullen vorkommen, intransitiv sind," *Math. Annalen*, vol. 52, p. 363 (December, 1898), where this restriction is dropped. Other proofs of the theorem have since been given by Frobenius, Burnside and Schur, a generalization by Loewy.

† "Bestimmung aller ternären und quaternären Collineationsgruppen, welche mit symmetrischen und alternierenden Buchstabenvertauschungsgruppen holoedrisch isomorph sind," *Math. Annalen*, vol. 51, p. 253 (November, 1897).

then the expression

$$(F^1, F^2, \cdots, F^n) = |\,a_{ik}\,|^{-\frac{1}{2}} \frac{\partial(F^1, F^2, \cdots, F^n)}{\partial(x_1, x_2, \cdots, x_n)}$$

is also an invariant of A. By means of this theorem it is easy to build up invariants for those special differential quantics A which are squares of complete differentials

$$\sum a_{ik} dx_i dx_k = (df)^2 = (f_1 dx_1 + \cdots + f_n dx_n)^2,$$

$$a_{ik} = f_i f_k, \quad f_i = \frac{\partial f}{\partial x_i},$$

and for systems of such quantics

$$A^1 = (df^1)^2, \; A^2 = (df^2)^2, \; \cdots;$$

for instance*

$$(f^1, f^2, \cdots, f^n)^2, \quad (u, f^2, \cdots, f^n)^2, \quad (u, f^2, \cdots, f^n)(v, f^2, \cdots, f^n),$$

etc., where u, v are arbitrary functions of $x_1, x_2, \cdots, x_n$, are such invariants.

But every such invariant is at the same time an invariant for the *general* quantic A, provided it is homogeneous and of the second degree in the quantities f^1_i, in the quantities f^2_i, etc. Thus the expression $(df)^2$ may be used as a *symbol for the quadratic differential* A in the same sense in which a^n_x is used as a symbol for the general binary n-ic in the ordinary theory of invariants, and $(df^1)^2$, $(df^2)^2$, . . ., etc., as *equivalent symbols* for A, if A^1, A^2, . . . coincide. Thus for instance the above invariants, considered as symbolic expressions, have a real meaning for the general quantic A, viz., they are respectively equal to $n\,!$, the "first differential parameter" $\Delta_1 u$, and the "intermediate differential parameter" $\nabla(u, v)$.

The remark that the total differential

$$dF = F_1 dx_1 + \cdots + F_n dx_n \quad (= F_x)$$

of an invariant F is a covariant of A, leads to symbolic expressions for covariants; for instance the symbolic expression

$$(f^1, f^2, \cdots, f^n)\,(u, f^2, \cdots, f^n)_x f^1_x$$

represents a quadratic covariant.

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* Maschke uses the more condensed notation $(f)^2$, $(uf)^2$, $(uf)(vf)$.

A symbolic expression which contains higher derivatives of the symbolic function f does not always have a real meaning, but only if these derivatives occur in certain combinations; the simplest combination of this kind, $f_i f_{kl}$ is equal to Christoffel's triple index symbol of the first kind

$$f_i f_{kl} = \begin{bmatrix} kl \\ i \end{bmatrix},$$

Hence it follows, for instance, that the symbolic expression

$$[(u, f^2, \cdots, f^n), f^2, \cdots, f^n]$$

has a real meaning; it is equal to $(n-1)!$ into the "second differential parameter" $\Delta_2 u$.

Maschke first explained his method, in a preliminary way, for the special case $n = 2$ in the paper "A new method for determining the differential parameters and invariants of quadratic differential quantics."*

In a later paper, "A symbolic treatment of the theory of invariants of quadratic differential quantics of n variables," † Maschke gives an elaborate and systematic development of his method for the general case of n variables. Very simple symbolic expressions are obtained for Christoffel's triple index symbol of the second kind, $\left\{ {kl \atop i} \right\}$, his quadruple index symbol $(ik\,rs)$, the quadrilinear covariant G_4, etc., and finally the three simplest general invariants (proper) of the second order are determined.

The great power of Maschke's symbolic method can best be seen from the applications which have been made to differential geometry in three-dimensional space by A. W. Smith in his doctor dissertation,‡ "The symbolic treatment of differential geometry," and to various problems of differential geometry in hyperspace by Maschke himself in his last two papers, "Differential parameters of the first order" § and "The Kronecker-Gaussian curvature of hyperspace." ‖

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* *Transactions of the American Mathematical Society*, vol. 1, p. 197 (April, 1900).

† *Ibid.*, vol. 7 (1907), p. 33.

‡ *Ibid.*, vol. 7. An interesting connection between Maschke's symbolic method and vector analysis has been established by L. Ingold in his doctor dissertation, "Vector interpretation of symbolic differential parameters."

§ *Ibid.*, vol. 7, p. 69 (dated September, 1905).

‖ *Ibid.*, vol. 7, p. 81 (dated September, 1905).

In the first of these papers he determines all directions of a space R_μ of μ dimensions through a point P of R_μ, which are normal to all the directions of a second space R_λ containing the point P, when R_μ and R_λ are both immersed in a general space R_n whose arc element is given by

$$ds^2 = \Sigma a_{ik} dx_i dx_k.$$

In the second paper he develops the symbolic expression for Kronecker's extension of Gauss's curvature of a surface to a space of n dimensions R_n, immersed in a euclidean space of $n+1$ dimensions, in terms of the symbols of the differential quantic

$$ds^2 = \Sigma a_{ik} du_i du_k \qquad (i, k = 1, 2, \cdots, n),$$

which defines the arc element of R_n, and then generalizes his results to a space R_λ of λ dimensions immersed in a general space of n dimensions, and defined by $n - \lambda$ relations

$$U' = \text{const}, \quad U'' = \text{const}, \quad \cdots, \quad U^{(n-\lambda)} = \text{const}.$$

Further applications are sure to follow, and Maschke's symbolic method will doubtless play in the theory of differential quantics a rôle similar to that of the ordinary symbolic method of Aronhold and Clebsch in the theory of algebraic quantics, thus, together with his contributions to the theory of linear substitution groups, securing him a permanent place in the history of mathematics.

OSKAR BOLZA.

FREIBURG, I. B.,
August 8, 1908.

NOTES.

AT the annual meeting of the AMERICAN MATHEMATICAL SOCIETY, to be held in the last week of December, President H. S. WHITE will deliver his retiring address, the subject of which will be "Bezout's theory of resultants and its influence on geometry."

THE second regular meeting of the Southwestern Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Kansas, Lawrence, Kansas, on Saturday, November 28. Titles and abstracts of papers to be presented at this meeting should be in the hands of the Secretary of the Section, Professor O. D. Kellogg, University of Missouri, Columbia, Mo., on or before November 14.

The concluding (October) number of volume 9 of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY contains the following papers: "Boundary values and expansion problems of ordinary linear differential equations," by G. D. BIRKHOFF; "An application of the form problems associated with certain Cremona groups to the solution of equations of higher degree," by A. B. COBLE; "On the differential equations of the equilibrium of an inextensible string," by E. B. WILSON; "The properties of curves in space which minimize a definite integral," by MAX MASON and G. A. BLISS; "The second derivatives of the extremal integral," by ARNOLD DRESDEN; "Sets of metrical hypotheses for geometry," by R. L. MOORE; "Notes and errata, volume 9."

The concluding (October) number of volume 30 of the *American Journal of Mathematics* contains the following papers: "On a group of transformations which occurs in the problem of several bodies," by E. O. LOVETT; "Normal curves of genus 6, and their groups of birational transformations," by VIRGIL SNYDER; "On the range of birational transformations of curves of genus greater than the canonical form," by VIRGIL SNYDER; "A set of assumptions for projective geometry," by OSWALD VEBLEN and J. W. YOUNG; "On the pentastroid," by R. P. STEPHENS; "A table of the values of m corresponding to given values of $\phi(m)$," by R. D. CARMICHAEL.

THE seventy-eighth meeting of the British association for the advancement of science was held at Dublin, September 2–9, 1908, under the presidency of Professor F. DARWIN, Dr. W. N. SHAW being chairman of section A, mathematics and physics. Twenty-four papers, mainly on physical or astronomical subjects, were read, many of which took the nature of symposia. The next meeting will be held at Winnipeg, Canada, opening August 25, 1909, under the presidency of Professor J. J. THOMSON.

THE thirty-seventh annual meeting of the French association for the advancement of science was held at Clermont-Ferrand, August 3 to 10, 1908, under the presidency of Professor P. APPELL, dean of the faculty of sciences of the University of Paris. The following papers were read before Sections one and two, mathematics, astronomy, geodesy and mechanics. By E. N. BARISIEN, "Solution of the equation of the third degree"; by E. BOREL, "Instruction in mathematics given by the universities"; by A. BOUTIN, "On a certain group of numbers"; by A. BOUTIN, "Development of $\sqrt{N}$ in a continued fraction, and the solution of Fermat's equations"; by E. LEBON, "Investigation of prime factors of large numbers"; by A. PELLET, "On equations having only real roots"; by P. APPELL, "On a theorem relative to the initial displacement of a system without friction"; by A. GÉRARDIN, "Study of numbers in affinity"; by A. GÉRARDIN, "On the solution of the equation $x^p \pm ay^2 = A^2$ in terms of positive integers"; by A. GÉRARDIN, "General solutions of the equation $ax^2 + by^2 = cz^2 + dt^2$"; by J. RICHARD, "Some points in the philosophy of mathematics"; by J. RICHARD, "Instruction in astronomy"; by T. ROUSSEAU, "Elementary geometry based on the group of displacements"; by J. WELSCH, "On homographic correspondence and its application to the solution of a large number of problems"; by H. CHRÉTIEN, "The comet Daniel 1907-d and its spectrum"; by H. CHRÉTIEN, "A new model for spectroheliography"; by L. LIBERT, "A catalogue of twenty-five meteors"; by E. BELOT, "Essay on cyclonic cosmogeny."

The next meeting of the association will be held at Lille in 1909, sections one and two under the chairmanship of Professor E. LEBON.

THE annual list of American doctorates published in *Science* presents for the academic year 1907–1908 360 names, of which 184 are credited to the sciences. The following 22 successful candidates offered mathematics as major subject (the titles of the theses are appended): E. G. BILL, Yale, "An a priori existence theorem for three dimensions in the calculus of variations"; R. L. BÖRGER, Chicago, "On the determination of ternary linear groups in the Galois field of order p^2"; G. G. CHAMBERS, Pennsylvania, "The groups of isomorphisms of the abstract groups of order p^2q"; G. M. CONWELL, Princeton, "The 3-space projective geometry (3, 2) and its group"; Miss E. B. COWLEY, Columbia, "Plane curves of the eighth order having two four-fold points with distinct tangents and no other point singularities"; C. F. CRAIG, Cornell, "On a class of hyperfuchsian functions"; F. J. HOLDER, Yale, "Multiple series;" L. INGOLD, Chicago, "Vector interpretation of symbolic parameters"; F. IRWIN, Harvard, "The invariants of linear differential expressions"; N. J. LENNES, Chicago, "Curves in non-metrical analysis situs, with applications to the calculus of variations and differential equations"; J. J. LUCK, Virginia, "The structures of the non-integrable groups of seven parameters"; E. B. LYTLE, Yale, "Multiple series over iterable fields"; C. N. MOORE, Harvard, "On the theory of convergence factors and some of its applications"; F. W. OWENS, Chicago, "The introduction of ideal elements and construction of projective n-space in terms of a plane system of points involving order and Desargues's theorem"; E. C. F. PHILLIPS, Johns Hopkins, "On the pentacardioid;" J. H. SCARBOROUGH, Vanderbilt, "The computation of the orbit of a planet"; Miss M. E. SINCLAIR, Chicago, "On a compound discontinuous solution connected with the surface of revolution of minimum area"; H. L. SLOBIN, Clark, "On plane quintic curves"; Miss A. L. VAN BENSCHOTEN, Cornell, "The birational transformations of algebraic curves of genus 4"; N. R. WILSON, Chicago, "Isoperimetric problems which are reducible to non-isoperimetric problems"; H. C. WOLFF, Wisconsin, "The continuous plane motion of a liquid bounded by two right lines"; Miss E. R. WORTHINGTON, Yale, "Some theorems on surfaces."

The number of American doctorates in mathematics for each year of the last decade is 13, 11, 18, 8, 7, 14, 21, 11, 13, 22.

THE Italian scientific society has awarded its gold medal for his researches in mathematics to the late professor G. FICCIATI, of the University of Bologna.

PROFESSOR C. SOMIGLIANA, of the University of Turin, has been elected member of the Accademia dei Lincei.

PROFESSOR E. ALMANSI, of the University of Pavia, has been elected corresponding member of the institute of Lombardy.

PROFESSOR F. SEVERI, of the University of Padua, has been elected corresponding member of the Institute of Venice.

PROFESSOR V. VOLTERRA, of the University of Rome, has been elected foreign associate of the academy of sciences of Halle.

DR. R. BONOLA has been appointed docent in projective and and descriptive geometry at the University of Pavia.

DR. M. N. VANECEK, of the Bohemian technical school at Prague, has been promoted to an associate professorship of mathematics.

PROFESSOR W. FENSSNER, of the University of Marburg, has been made honorary professor.

PROFESSOR R. FRICKE, of the technical school at Brunswick, has declined a call to the professorship of mathematics at the technical school at Hanover.

PROFESSOR H. A. LORENTZ, of the University of Leiden, has been elected foreign knight of the order pour le mérite in sciences and arts.

PROFESSOR H. POINCARÉ has been elected honorary member of the physico-chemical society of Erlangen.

MR. R. F. SCOTT has been elected master of St. John's College, Cambridge, in succession to the late Dr. C. TAYLOR.

PROFESSOR M. LACOMBE, of the technical school at Zürich, has been appointed professor of geometry at the University of Lausanne, as successor to the late Professor Joly.

DR. H. HUSSON has been appointed professor of mechanics at the University of Caen.

PROFESSOR M. DULAC, of the University of Poitiers, has been appointed professor of mathematics at the scientific school of Algiers.

DR. M. BOURGEOIS has been appointed professor of astronomy and geodesy at the Ecole polytechnique of Paris, to succeed Professor H. POINCARÉ, who has retired.

PROFESSOR A. S. CHESSIN, of Washington University, St. Louis, has been granted a year's leave of absence.

MR. F. H. HODGE has been appointed professor of mathematics at Parsons College, Fairfield, Iowa.

AT the University of North Carolina, Professor A. HENDERSON has been promoted to a full professorship of mathematics. Mr. G. K. G. HENRY and Mr. J. C. HINES have been appointed instructors in mathematics.

AT the University of Kansas Mr. G. W. HARTWELL, of Columbia University, has been appointed assistant professor of mathematics; Dr. P. WERNICKE, of Washington University, Mr. M. GABA and Mr. C. A. PIERCE have been appointed instructors in mathematics.

AT the Worcester Polytechnic Institute Professor L. L. CONANT has been promoted to the head professorship of mathematics; Professor A. D. BUTTERFIELD, of the University of Vermont, has been appointed assistant professor of mathematics; Mr. J. J. BULLARD and Mr. J. H. REDFIELD have been appointed instructors in mathematics.

PROFESSOR G. A. BLISS, of Princeton University, has been appointed associate professor of mathematics at the University of Chicago.

PROFESSOR T. E. MCKINNEY, of Wesleyan University, has been appointed professor of mathematics at the University of South Dakota.

MR. N. C. RIGGS has been appointed assistant professor of mathematics at the Carnegie Technical School at Pittsburg.

DR. W. B. STONE and MR. J. B. CLEVELAND have been appointed instructors in mathematics at the University of Michigan.

MR. H. F. MACNEISH has been appointed instructor in mathematics at Princeton University.

DR. R. L. MOORE, of Princeton University, has been appointed instructor in mathematics at Northwestern University.

DR. E. B. LYTLE has been appointed instructor in mathematics at the University of Illinois.

MR. C. W. COBB has been appointed instructor in mathematics at Amherst College.

RECENT catalogues of second hand mathematical books: Conrad Skopnik, Berlin N. W. 7, Prinz Louis Ferdinandstrasse 1, catalogue no. 39, 95 titles on mathematics. — Theodor Ackermann, Munich, 10 Promenadeplatz, catalogue no. 572, 903 titles, partly mathematical. — J. Schweitzer, Munich, Lenbachplatz 1, catalogue no. 45, 76 titles. — Eduard Beyer's Nachfolger, Vienna, Schottengasse 7, catalogue no. 48, exact sciences, 2250 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Ahrens (W.). See Stäckel (P.).

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THE SEPTEMBER MEETING OF THE SAN FRANCISCO SECTION.

THE fourteenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of California, Saturday, September 26. The following eleven members of the Society were present: Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor J. H. McDonald, Professor H. C. Moreno, Professor C. A. Noble, Professor T. M. Putnam, Mr. H. W. Stager, Mr. J. D. Suter.

Professor D. N. Lehmer was elected to succeed Professor Hoskins as chairman of the Section. The present Secretary was reëlected, and Professors Blichfeldt and McDonald were named to serve with the secretary on the program committee, Dr. Blichfeldt being designated as chairman of the committee.

The two meetings in 1909 will be held on February 27, at Stanford University, and on September 25, at the University of California.

The following papers were read at this meeting:

(1) Professor H. F. BLICHFELDT: "A theorem on simple groups."

(2) Professor J. H. McDONALD: "On Fourier series and expansions in spherical harmonics."

(3) Professor H. C. MORENO: "On a class of ruled loci."

(4) Dr. ARTHUR RANUM: "The general term of a recurring series."

(5) Professor C. A. NOBLE: "Necessary conditions that three or more partial differential equations of the second order shall have common solutions."

Abstracts of these papers are given below in the order of their presentation.

1. Professor Blichfeldt's paper is in substance as follows: Let F be a group of order $p^\lambda n$, p being prime to n, containing a Sylow subgroup G of order p^λ. This has a subgroup G_1 of order $p^{\lambda-1}$. Let S be one of the substitutions of lowest order contained in G but not in G_1. Then, either F will contain an invariant subgroup of index p, or there is in G a subgroup K

containing at least every substitution A of G for which $SAS^{-1}A^{-1}$ is an invariant substitution of G_1, — this group K being transformed into itself by a substitution T of H, though T does not transform S into itself. The order of T is prime to p. Further properties of K were stated; e. g., $K = G$ when the latter contains no substitutions of order p^2. The types of certain Sylow subgroups G contained in a simple group H were shown to be restricted by this theorem. Cf. Frobenius, "Über auflösbare Gruppen, V," *Berliner Sitzungsberichte*, 1901, pages 1324–1329.

2. Professor McDonald's paper discusses the application of Poisson's integral to the summation of a Fourier series and investigates Laplace's method of expansion in spherical harmonics in a similar way.

3. The general problem of finding, in a space of n dimensions, the locus of all the lines meeting n $(n - 2)$-flats was presented by Professor Moreno at a previous meeting of this section. In the present paper, the same problem for $n = 4$ is treated more in detail. By a special choice of coordinates the equation of the 3-spread was shown to be $\Sigma xyz + n^2\Sigma x = 0$. This is Segre's cubic spread. In this form of the equation, many properties of the spread are readily deduced.

4. Let $u_0 + u_1 + u_2 + \cdots$ be a recurring series of the nth order, in which every term after the nth is expressed as a linear function of the n preceding terms by means of the formula $u_m = a_1u_{m-1} + \cdots + a_nu_{m-n}$. By successive applications of this formula the general term is expressible as a linear function of the first n terms, and a rational integral function of the n constants of the scale, say $u_m = F_m(u_0, \cdots, u_{n-1}; a_1, \cdots, a_n)$. Nevertheless, the classical method of finding it involves the irrational process of decomposing the generating function of the series into partial fractions. In this paper Dr. Ranum gives a rational method and determines the explicit form of the function F_m. As an application, the mth power of a linear substitution in n variables is expressed in terms of its first $n - 1$ powers.

5. Following a suggestion made by Hilbert in his lectures in 1900, and which was carried out in some detail by Yoshiye

in volume 57 of the *Annalen*, Professor Noble extends the results of Yoshiye by finding the necessary conditions that three or more equations of the form

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}\right) = 0$$

shall have solutions in common.

W. A. MANNING,
Secretary of the Section.

NOTE ON STATISTICAL MECHANICS.

BY PROFESSOR EDWIN BIDWELL WILSON.

(Read before the American Mathematical Society, September 11, 1908.)

IN developing the elements of statistical mechanics it is customary though by no means essential to remark the analogy between that subject and hydromechanics.* The analogy arises primarily through the fact that the equation of continuity of hydrodynamics,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where u, v or u, v, w are the velocities in the fluid in two or in three dimensions, exists in the form

$$(1) \qquad \frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} = 0 \quad \text{or} \quad \sum_{i=1}^{i=n} \frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial \dot{q}_i}{\partial q_i} = 0$$

for dynamical systems regulated by the hamiltonian canonical equations

$$(2) \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2, \cdots, n);$$

and it is further exemplified by connection of the Boltzmann-Larmor hydrodynamical interpretation of Jacobi's last multiplier with the principle of conservation of extension in phase.† The object of this note is to comment upon the analogy in question.

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* Jeans, The Dynamical Theory of Gases, p. 62. Gibbs, Elementary Principles in Statistical Mechanics, p. 11.

† Compare Whittaker, Analytical Dynamics, p. 272, and Gibbs, loc. cit., p. 29.

Let the $2n$ quantities $q_1, q_2, \cdots, q_n, p_1, p_2, \cdots, p_n$ be interpreted as rectangular coordinates in a space of $2n$ dimensions. And suppose that an ensemble of systems which satisfy (2) be imagined as distributed in that space with a density in phase D or with a probability in phase P which are functions of p_i, q_i, and the time t. Then, by the same reasoning as in hydromechanics, the density D or the probability P will satisfy the equation

$$\frac{dP}{dt} + P\sum_{i=1}^{i=n}\left(\frac{\partial \dot{p}_i}{\partial p_i} + \frac{\partial q_i}{\partial q_i}\right) = \frac{dP}{dt} = 0.$$

As the condition for statistical equilibrium is that the density shall be independent of the time, it follows that for statistical equilibrium

$$\frac{\partial P}{\partial t} = -\sum_{i=1}^{i=n}\left(\frac{\partial P}{\partial p_i}\dot{p}_i + \frac{\partial P}{\partial q_i}\dot{q}_i\right) = 0.$$

These results are obtained with full discussion by Gibbs, loc. cit.

Suppose that the systems considered are specified by a single coordinate q and the momentum p. The connected hydrodynamical problem thus becomes two-dimensional. Let it be asked whether the fluid which represents the ensemble of systems is more fully comparable with the ordinary fluid of hydrodynamics than is implied by the existence of an analogue for the equation of continuity. In the first place it should be noted that the motion of the fictitious fluid is given by equations relating to its velocities, that is, by

$$\dot{q} = \frac{\partial H}{\partial p} = u(p, q), \quad \dot{p} = \frac{\partial H}{\partial q} = v(p, q).^*$$

If the velocities of a particle in ordinary dynamics be given, a differentiation with respect to the time will suffice to determine the forces. If the velocities in a continuous medium are given, the problem is not so simple. For suppose that

$$u = u(x, y) \quad \text{and} \quad v = v(x, y)$$

are the velocities in an ordinary fluid. Then

---

* For convenience and definiteness it will be assumed throughout this note that H does not depend explicitly upon the time and moreover that H may be decomposed into the sum of the kinetic and potential energies.

$$\frac{du}{dt} = \frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial t} = X - \frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$\frac{dv}{dt} = \frac{\partial v}{\partial x}u + \frac{\partial V}{\partial y}v + \frac{\partial V}{\partial t} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y},$$

where ρ is the density, p is the pressure, and X and Y are the components of the impressed forces per unit mass. It therefore appears that ρ and p may be assigned at pleasure and X, Y determined to suit.

To remove this indeterminateness, it might be required that the forces X, Y be derivable from a potential. Then $\dot{u}$ and $\dot{v}$, known functions of the variables, would afford the two equations

$$\dot{u} = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \dot{v} = -\frac{\partial V}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y}.$$

If ρ be constant, as in the dynamics of liquids, the problem is now impossible unless

$$\dot{u}dx + \dot{v}dy$$

is a perfect differential. On the other hand if ρ be not constant but a known function of the variables, the expression of the fact that

$$\left(\dot{u} + \frac{1}{\rho}\frac{\partial p}{\partial x}\right)dx + \left(\dot{v} + \frac{1}{\rho}\frac{\partial p}{\partial y}\right)dy$$

be a perfect differential will afford a partial differential equation from which p may be determined. It should be remembered, however, that in the case of any real fluid, there is generally assumed an equation $F(p, \rho, T) = 0$ connecting the pressure, density, and temperature. The temperature is generally disregarded for purposes of hydrodynamics and it is assumed that $F(p, \rho) = 0$. The determination of p by the method suggested above would not generally give a result in which p was a function of ρ. In fact if p is a function of ρ, the equations may be written as

$$\dot{u} = -\frac{\partial V}{\partial x} - \frac{\partial \bar{p}}{\partial x}, \quad \dot{v} = -\frac{\partial V}{\partial y} - \frac{\partial \bar{p}}{\partial y}, \quad \bar{p} = \int \frac{dp}{\rho}$$

and the problem would again generally be impossible.

With these points in connection with the purely hydrodynamical problem in mind we may return to the matter of statistical mechanics and ask in the first place what are the conditions that there should be a velocity potential for the motion of the fictitious fluid. If, when there is only one variable,

$$\dot{q}dq + \dot{p}dp = \frac{\partial H}{\partial p}dq - \frac{\partial H}{\partial q}dp$$

is a perfect differential, the expression

$$\Delta H = \frac{\partial^2 H}{\partial p^2} + \frac{\partial^2 H}{\partial q^2} = 0.$$

Now by hypothesis, H is of the second order in p. Hence

$$\frac{\partial^3 H}{\partial p^3} + \frac{\partial^3 H}{\partial p \partial q^2} = 0 \quad \text{gives} \quad \frac{\partial^3 H}{\partial p^3} = \frac{\partial^3 H}{\partial p \partial q^2} = 0.$$

The differentiation with respect to p and q to find the derivatives of the fourth order shows at once that

$$\frac{\partial^4 H}{\partial p^4} = \frac{\partial^4 H}{\partial p^3 \partial q} = \frac{\partial^4 H}{\partial p^2 \partial q^2} = \frac{\partial^4 H}{\partial q^4} = 0.$$

Hence H is a polynomial of at most the third degree in p and q.

If there had been more variables, say two, the conditions that

$$\frac{\partial H}{\partial p_1}dq_1 - \frac{\partial H}{\partial q_1}dp_1 + \frac{\partial H}{\partial p_2}dq_2 - \frac{\partial H}{\partial p_2}dp_2$$

be a perfect differential would be

$$\frac{\partial^2 H}{\partial p_1^2} + \frac{\partial^2 H}{\partial q_1^2} = 0, \qquad \frac{\partial^2 H}{\partial p_2^2} + \frac{\partial^2 H}{\partial q_2^2} = 0,$$

$$\frac{\partial^2 H}{\partial p_1 \partial p_2} + \frac{\partial^2 H}{\partial q_1 \partial q_2} = 0, \quad \frac{\partial^2 H}{\partial p_1 \partial q_2} = \frac{\partial^2 H}{\partial p_2 \partial q_1} = 0,$$

and in a similar manner would show that H must be a polynomial of at most the third degree in p_i and q_i jointly. The result is the same for any number of variables. Hence it appears that only in a very restricted type of cases can there exist a velocity potential for the fictitious fluid, namely, when H, being of the second degree in p_i, is at most of the third degree in p_i and q_i jointly, and even in this case there are still certain relations to be satisfied by the coefficients of the polynomial H. In such cases only would a velocity potential exist for the fictitious motion.

Let it now be asked whether the equations

$$\dot{q} = -\frac{\partial V}{\partial q} - \frac{1}{P}\frac{\partial \Pi}{\partial q}, \quad \dot{p} = -\frac{\partial V}{\partial p} - \frac{1}{P}\frac{\partial \Pi}{\partial p},$$

where P is the probability in phase and Π is a fictitious pressure for the fictitious fluid, can be satisfied. In the first place it may be noted that

$$\ddot{q} = \frac{d}{dt}\frac{\partial H}{\partial p} = \frac{\partial^2 H}{\partial q \partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2}\frac{\partial H}{\partial q} = -J\left(H, \frac{\partial H}{\partial p}\right),$$

$$\ddot{p} = -\frac{d}{dt}\frac{\partial H}{\partial q} = -\frac{\partial^2 H}{\partial q^2}\frac{\partial H}{\partial p} + \frac{\partial^2 H}{\partial q \partial p}\frac{\partial H}{\partial q} = J\left(H, \frac{\partial H}{\partial q}\right),$$

where J denotes the jacobian with respect to q and p of the quantities in the parenthesis. The equations to be satisfied are therefore

$$-J\left(H, \frac{\partial H}{\partial p}\right) + \frac{1}{P}\frac{\partial \Pi}{\partial q} = -\frac{\partial V}{\partial q},$$

$$J\left(H, \frac{\partial H}{\partial q}\right) + \frac{1}{P}\frac{\partial \Pi}{\partial p} = -\frac{\partial V}{\partial p}.$$

The equation obtained by differentiating these expressions respectively with respect to p and q and subtracting is seen at once to be simply

$$(3) \qquad J(H, \Delta H) = J(\Pi, P^{-1}).$$

This is a partial differential equation of the first order and first degree in Π which will in general suffice to determine Π when P is known.

The form of (3) is such that one interesting theorem is obvious: The necessary and sufficient condition that there may be a functional relation $F(\Pi, P) = 0$ between the pressure and density in the fictitious fluid is that the function H satisfy the relation

$$\Delta H = \frac{\partial^2 H}{\partial p^2} + \frac{\partial^2 H}{\partial q^2} = \Phi(H),$$

where Φ is any function of H. In this case $\dot{q}\,dq + \bar{p}\,dp$ is a perfect differential and the complete integration of the differential may be found.

$$\int\left(\frac{\partial^2 H}{\partial q\partial p}\frac{\partial H}{\partial p} - \frac{\partial^2 H}{\partial p^2}\cdot\frac{\partial H}{\partial q}\right)dq + \left(\frac{\partial^2 H}{\partial q\partial p}\frac{\partial H}{\partial q} - \frac{\partial^2 H}{\partial q^2}\frac{\partial H}{\partial p}\right)dp =$$

$$\frac{1}{2}\left[\left(\frac{\partial H}{\partial p}\right)^2 + \left(\frac{\partial H}{\partial q}\right)^2\right] - \int\Phi(H)\,dH = -V - \overline{\Pi},$$

where

$$\overline{\Pi} = \int\frac{d\Pi}{P} = \int\frac{d\Pi}{dH}\frac{dH}{P}.$$

It therefore appears natural to take the functions V and Π as

$$V = -\tfrac{1}{2}\left[\left(\frac{\partial H}{\partial p}\right)^2 + \left(\frac{\partial H}{\partial q}\right)^2\right], \quad \Pi = \int P\,\Phi(H)dH.^*$$

Here P may be any function of H which satisfies the necessary relation

$$\int\int P dp dq = 1 \quad \text{(within limits considered)},$$

although there is a large family of functions defined by such a general relation as $\Delta H = \Phi(H)$. The fact that H is necessarily quadratic in p suffices, as in the case of $\Delta H = 0$, to determine H except for a few arbitrary constants of integration. In fact, under the assumption that H is divisible into a kinetic and a potential energy,

$$H = \tfrac{1}{2}f(q)p^2 - \phi(q), \quad \Delta H = f(q) + \tfrac{1}{2}f''(q)p^2 - \phi''(q),$$

$$J(H, \Delta H) = \begin{vmatrix} \tfrac{1}{2}f'(q)p^2 - \phi'(q) & f(q)p \\ f'(q) + \tfrac{1}{2}f'''(q)p^2 - \phi'''(q) & f''(q)p \end{vmatrix} \equiv 0,$$

and hence

$$f(q)f'''(q) - f''(q)f'(q) = 0,$$

$$f(q)\left[f'(q) - \phi'''(q)\right] + f''(q)\phi'(q) = 0.$$

The first of these equations gives $f'' = kf$ and the second reduces to

$$f'(q) - \phi'''(q) + k\phi'(q) = 0.$$

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[*] This choice seems the simplest; but any function of H may be added to Π and a compensating change introduced in V.

The complete determination of f then gives the results

$$f = aq + b, \quad f = a\cos(\sqrt{-k}q + b), \quad f = ae^{\sqrt{k}q} + bc^{-\sqrt{k}q},$$

according as k is zero, negative or positive. The determination of $\phi(q)$ then calls for the solution of a linear equation with constant coefficients and with second member. The class of problems of mechanics which result from such a value of H is somewhat restricted. In case the coordinate q is so chosen that $f(q) = \text{const}$, $\phi'''(q) = 0$; and the potential energy is a quadratic in q — a very restricted but well known and frequently occurring type of problem.*

To return to the general case, $J(H, \Delta H) \neq 0$. Here the equation (3) will determine Π when P is given. As there is only one variable, there is only one invariant of the motion of any given one of the systems in the ensemble, and that invariant is H. For statistical equilibrium, it is therefore necessary that P shall be a function of H. As H and ΔH are functionally independent, they may be taken as the independent variables, and by the rules of operation with jacobians,

$$1 = \frac{J(\Pi, P^{-1})}{J(H, \Delta H)} = \begin{vmatrix} \dfrac{\partial \Pi}{\partial \Delta} & \dfrac{\partial P^{-1}}{\partial H} \\[2ex] \dfrac{\partial \Pi}{\partial \Delta H} & \dfrac{\partial P^{-1}}{\partial \Delta H} \end{vmatrix}$$

Hence for statistical equilibrium this result becomes merely

$$\frac{dP^{-1}}{dH}\frac{\partial \Pi}{\partial \Delta H} = -1, \quad \Pi = -\frac{dH}{dP^{-1}}\Delta H + f(H).\dagger$$

The expression for Π is determined except for an additive arbitrary function of H. No more complete determination could be expected because the fictitious potential V may take on an additive function of H.

For simplicity it may be assumed that the additive function $f(H)$ is omitted from Π. The determination of V is then straightforward and simple. In fact

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* It may be noted that if $\Delta H = 0$ in this case, the potential energy is $-\phi(q) = -\frac{1}{2}f(q)q^2 + mq + n$, where $f(q)$ is a constant necessarily positive. The force is therefore repulsive, and Gibbs's canonical distribution would be illusory. See Gibbs, loc. cit., p. 35.

† This expression which is of the form $\Pi = \phi(P)\Delta H + \psi(P)$ corresponds to the equation $F(p, \rho, T) = 0$ of ordinary fluids. In a way, therefore, it may be said that ΔH corresponds to T.

$$V = P^{-1}\Delta H \frac{dH}{dP^{-1}} - \frac{1}{2}\left[\left(\frac{\partial H}{\partial p}\right)^2 + \left(\frac{\partial H}{\partial q}\right)^2\right].$$

With this value of V and the value for Π obtained above the equations of the fictitious fluid become identical with those of a real fluid. The values of V and Π may further be written as

$$V = -\Delta H \frac{dH}{d\eta} - \frac{1}{2}\left[\left(\frac{\partial H}{\partial p}\right)^2 + \left(\frac{\partial H}{\partial q}\right)^2\right],$$

$$\Pi = e^{\eta}\Delta H \frac{dH}{d\eta}, \qquad \text{where } \eta = \log P$$

is the index of probability in phase. As e^{η} is necessarily positive, $\Delta H \cdot dH/d\eta$ must be always positive if Π is to be positive, as in the case of a real fluid. In the case of Gibbs's canonical distribution

$$P = e^{\frac{\psi - H}{\Theta}}, \qquad \eta = \frac{\psi - H}{\Theta}$$

and $d\eta/dH$ is negative. For simple vibrating systems ΔH is positive * and hence for these systems Π would be negative and represent a tension.

The difficulty that Π is frequently negative may be remedied in some cases by a proper choice of the additive function $f(H)$. This, however, complicates the formulas and is not particularly desirable. In fact if the fictitious medium tended to expand, the concept of statistical equilibrium would be difficult of application. On the whole, the hydrodynamic analogy does not appear to be as close and vital as could be desired when developed to a greater extent than is implied in the equation (1) of continuity, although tolerably satisfactory results have been obtained for the case $n = 1$. When the number of variables n is greater than one, the $2n$ equations

$$\ddot{q}_i = -\frac{\partial V}{\partial q_i} - \frac{1}{P}\frac{\partial \Pi}{\partial q_i}, \qquad \ddot{p}_i = -\frac{\partial V}{\partial p_i} - \frac{1}{P}\frac{\partial \Pi}{\partial p_i}$$

cannot be satisfied except under extremely restricted hypotheses concerning H. The conditions that the expression

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* For instance when $H = \frac{1}{2}p^2 + \frac{1}{2}n^2q^2 + kq^4$. If the system is that of harmonic vibrations, ΔH is constant, $J(H, \Delta H) = 0$, and the system comes under a previous type.

$$\Sigma\left(q_i + \frac{1}{P}\frac{\partial \Pi}{\partial q_i}\right)dq_i + \Sigma\left(p_i + \frac{1}{P}\frac{\partial \Pi}{\partial p_i}\right)dp_i = -dV$$

are too numerous to be satisfied by any functions P, Π, V. It is, however, possible that a medium, which satisfies the equation of continuity (1) and has equations more complicated than the hydromechanical type, some elastic medium, may be found to answer the purposes of the problem.

From a physical point of view, however, there is one matter which is of serious disadvantage in any hydrodynamical interpretation of the problem of statistical mechanics. That is the question of dimensions. If v denote the velocity in the fictitious fluid, then

$$v^2 = \dot{q}_1^2 + \dot{p}_1^2 + q_2^2 + \dot{p}_2^2 + \cdots$$

is without physical significance unless the dimensions of q_1, $q_2, \cdots, p_1, p_2, \cdots$ are all alike. But the very equations (2) defining q_i, p_i, give

$$[q_i] = \frac{[H]}{[p_i]}, \quad [p_i] = \frac{[H]}{[q_i]}, \quad [p_iq_i] = [H], \quad [p_i] = [q_i],$$

where the brackets are used as dimensional symbols. Hence the dimensions are

$$[p_i] = [q_i] = [m]^{\frac{1}{2}}\frac{[l]}{[t]}.$$

where m, l, t denote mass, distance, time. As p_i and q_i seldom, if ever, have such dimensions as are here required, v^2 is not a physical quantity. The same is true of ΔH. It is interesting to note that the equation of continuity is such that its dimensionality is always $[t]^{-1}$, as it should be.

The importance of the homogeneity of physical dimensions of formulas is so great and mathematical investigations and analysis carried on without due regard to dimensions are so seldom of physical importance, that despite the results obtained for $n = 1$, the above considerations suggest the improbability of finding any continuous medium which shall properly shadow forth the relations of statistical mechanics. Otherwise it would be tempting to try to find such media for the cases where $n \geqq 2$. That which replaces the detailed physical concept of a medium is the mathematical concept of integral invariants, and these are dimensionally homogeneous.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *August*, 1908.

ON CERTAIN CONSTANTS ANALOGOUS TO FOURIER'S CONSTANTS.

BY DR. C. N. MOORE.

THROUGH the kindness of Professor Landau my attention has been called to the fact that Theorem I of the article of the above title which appeared in the BULLETIN * may be obtained from a theorem due to Lerch † by means of a change of variable and a slight modification of the form of statement.

With regard to Theorem II of the same article I wish to state that the interval $a \leqq x \leqq b$ must be less than 2π in order that the proof there given may hold. However, the theorem is still true when the interval in question is equal to 2π, and may be proved very simply from a theorem due to Fejér. The latter has shown ‡ that if in an interval of length 2π a function is finite save for a finite number of points, and is integrable, the Fourier development corresponding to that function is summable to the value of the function § at every point of that interval at which the function is continuous. Under the conditions of Theorem II the Fourier development corresponding to $\psi(x)$ in the interval $a \leqq x \leqq b$ has each of its terms equal to zero and is therefore summable to the value zero at every point of that interval. However, we know from Fejér's theorem that the development is summable to the value of the function $\psi(x)$ at every point of the interval $a \leqq x \leqq b$ at which this function is continuous and hence $\psi(x)$ is zero at every such point.

CAMBRIDGE, MASS.,
June, 1908.

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* BULLETIN, ser. 2, vol. 14 (1908), p. 368.

† *Acta Mathematica*, vol. 27 (1903), p. 345.

‡ *Math. Annalen*, vol. 58 (1903–04), p. 59.

§ *I. e.*, the limit as n becomes infinite of the arithmetic mean of $s_1, s_2, \cdots, s_n$, where s_m represents the sum of the first m terms of the series, exists and is equal to the value of the function.

THE COLOGNE MEETING OF THE DEUTSCHE MATHEMATIKER-VEREINIGUNG.

THE eightieth convention of German naturalists and physicians was held at Cologne, September 20–26, 1908. Mathematics and astronomy together formed one of the thirty-one sections. The total attendance was about two thousand, of which the Deutsche Mathematiker-Vereinigung as a sub-division claimed nearly one hundred.

On the evening of September 20, an informal reception was given in the Bürgergesellschaft. A general session of all the divisions took place in the Gürzenich on the following morning. After greetings from representatives of the administration and from various societies in Cologne, two papers were read: (1) The achievements and influence of Albertus Magnus (one of the founders of the ancient Cologne University); (2) Motor balloons and flying machines.

The sessions of the Vereinigung were held in the Handels-Hochschule. Under the influence of President Klein, the society devoted the meetings largely to the problems of applied mathematics, thus acting as mentor for the Naturforscherversammlung. The following papers were read:

1. H. MINKOWSKI, Göttingen, "Space and time."
2. G. HAMEL, Brünn, "The foundations of mechanics."
3. E. H. TIMERDING, Strassburg, "The historical development of the concept of force."
4. P. STÄCKEL, Karlsruhe, "Remarkable motions of the top."
5. R. v. MISES, Brünn, "Problems of technical hydromechanics."
6. F. MÜLLER, Dresden, "Plans for the publication of the works of Leonhard Euler."
7. H. WIENER, Darmstadt, "On the geometry of the binary forms."
8. H. JUNG, Marburg, "Algebraic functions of two independent variables."
9. H. REISSNER, Aachen, "Scientific problems in the mechanics of flight."

Number 8 has already appeared in the *Comptes Rendus*; numbers 2 and 4 will soon be published in the *Mathematische Annalen*, and number 5 in the *Physikalische Zeitschrift*. Abstracts of the other papers are given below, the numbers corresponding to those in the list above.

1. Professor Minkowski discussed the complete revolution that has taken place in our conception of time and space, owing to the exact mathematical deductions from the latest investigations of physics. Lagrange has called physics a four-dimensional geometry because three dimensions of space, and one of time are introduced, and this definition seems to-day to be applicable in a deeper, unhoped-for sense. Minkowski shows that the remarkable hypothesis of H. A. Lorentz concerning the contraction of the electrons and the alleged contradiction between the newtonian mechanics and the modern theory of electricity can be completely accounted for if we assume that we live in a four-dimensional world of which the fourth dimension (time) may be neglected with a certain amount of freedom. The axiom that the velocity of matter cannot exceed that of light in ether plays here an important rôle. This new comprehension of the world as a sort of union of time and space makes possible great strides in the theory of electricity and magnetism, and requires finally a revision of all physical theories. That these revisions are capable of being carried out is due alone to the many-sided advances made by pure mathematics in the past century.

3. Professor Timerding's paper was an historical review of the mathematical conception of force with especial reference to the notions of Newton.

6. Professor Müller sketched the career and influence of Euler and outlined the plans that are being made for the publication of his works.

7. Professor Wiener continued his investigations in the geometric interpretation of binary forms, making use of polarity and reflection.

9. There are three chief properties of a flying machine, the capability of floating, the stability, and the power. The simplest cases of the first can be solved by theoretical hydrodynamics, but the more difficult must be experimentally determined. The problem of the most favorable shape of the planes and the most advantageous angle of motion has not been systematically treated since the time of Lilienthal, and our knowl-

edge regarding the position of the resulting pressure is very incomplete. It is shown that the problem of stability must be treated as a case of oscillation about a state of stationary motion. The vibration in the plane of motion may be regarded as consisting of two components, of which one however is so damped that it does not come into practical computation. In a machine consisting of parallel surfaces the presence of horizontal tail surfaces aids in lessening the oscillation, but they are not economical when the power comes into consideration. Models are not helpful in stability experiments because the theory of dimensions is not applicable. A strictly hydrodynamical treatment of the propeller problem is not at present possible, but the application of the laws of oblique motion of thin plates produces a good propeller. Professor Reissner has found also that the propeller which he constructed in accordance with his solution of the problem by means of the calculus of variations works to good advantage, and his researches in coöperation with Professor Junkus are to be continued on a larger scale.

The business session of the Deutsche Mathematiker-Vereinigung was held on the afternoon of the 22nd. The report showed that the present membership is about seven hundred. The executive committee for the year 1908–09 is as follows: M. Krause, president; A. Krazer, secretary; A. Ackermann-Teubner, treasurer; A. Gutzmer, editor of the *Jahresbericht;* A. Schönflies, F. Klein, F. Engel, and F. Schur. As German representatives in the international committee on mathematical instruction to report to the congress in Cambridge (1912), F. Klein and P. Treutlein were chosen. The society voted the sum of five thousand francs in aid of the publication of the works of Euler undertaken by the Swiss scientific societies.

Professors Bolza, of Chicago, Field, of Michigan, and Richardson, of Brown were the only Americans present. On the evening of the 21st the visitors were entertained in the zoological gardens by the scientific societies of Cologne, and on the following evening a banquet was held in the Gürzenich. On Wednesday evening the theatre and opera gave special performances to which the members were invited. The reception given by the city took place on Thursday evening, and on Saturday four excursions to various points of interest in the vicinity of Cologne were arranged. The next meeting will be held in Königsberg.

R. G. D. Richardson.

GOURSAT'S COURS D'ANALYSE.

Cours d'Analyse mathématique. By ÉDOUARD GOURSAT, Professeur à la Faculté des Sciences de Paris. Vol. II. *Theory of Analytic Functions; Differential Equations, Total and Partial; Elements of the Calculus of Variations.* Paris, Gauthier-Villars, 1905. 8vo, pp. vi + 640.

THE second and last volume of Professor Goursat's treatise on the elements of analysis is devoted to the theory of functions of one and of several complex variables, pages 1–305, and to differential equations, total and partial, pages 306–589, concluding with a chapter of forty-two pages on the calculus of variations. Like the first volume, the book is characterized by the vital relation of the choice of material and the mode of treatment to the analysis of the present time. It is the product of a man whose contributions to analysis are of value and who wishes to give to his students the tools he has found to be useful. The procedure that is simplest in practice is given the most prominent place in the presentation, and the main theorems of the subject treated are set in a strong light. On the other hand, a wide range of topics is treated, an outline of a whole theory being sometimes given well on in a chapter as an application of the foregoing principles. When this is done, the author shows good judgment in what he expects of his readers. It is reasonable to leave to the student at this stage the details of proofs which follow general well-defined lines, provided that the framework is so constructed that he can see what is required. The tact which the author displays in this regard makes the book a serviceable one for advanced students to read by themselves and to report on to their university teacher.

The volume begins with an introductory chapter of seventy-four pages, Chapter XIII, on analytic functions of a complex variable, in which both single and multiple-valued functions are treated, the elementary functions being defined in a natural manner for complex values of the argument. The evaluation of integrals of the type

$$\int R(x)dx, \quad \int R(\sin x, \cos x)dx,$$

where R is a rational function, is taken up, and the subjects of infinite series and products, conformal mapping both of plane and of curved surfaces, with applications to Mercator's chart and stereographic projection and to the transformation of an anchor ring on a rectangle, and families of isothermals, are set forth. In the twenty-five examples that conclude the chapter a number of properties of linear transformations of a complex variable are brought out.

Chapter XIV is devoted to an exposition of Cauchy's theory of functions, with applications to singular points and to Weierstrass's theorems about series of analytic functions. Lagrange's and Burmann's series are treated, the hyperelliptic integrals are studied with reference to questions of periodicity, and a proof that the ratio of the moduli of periodicity of an elliptic integral of the first kind cannot be a real quantity is given, by the aid of Weierstrass's law of the mean for complex integrals, without breaking the integral up into its real and its pure imaginary part. Applications to spherical harmonics appear repeatedly.

Chapter XV, on single-valued functions, contains the theorems of Weierstrass and Mittag-Leffler. The development of cot x is obtained by the method of residues, the contour of integration being a rectangle. Fifty-eight pages are then devoted to the theory of doubly periodic functions and their inverse, with applications to algebraic curves of deficiency unity.

Chapter XVI, on analytic continuation, and Chapter XVII, on analytic functions of several variables, conclude the part of the volume devoted to the theory of functions. The definition of an analytic function of two independent variables, given on page 264, is as follows: Let A and A' be two regions of the z-plane and the z'-plane respectively and let $f(z, z')$ be uniquely defined for each pair of values z, z' belonging to these regions. If $f(z, z')$ is continuous and each partial derivative $\partial f/\partial z$, $\partial f/\partial z'$, exists at each such point (z, z') — *i. e.*, if each of the ratios

$$\frac{f(z+h, z')-f(z, z')}{h}, \qquad \frac{f(z, z'+k)-f(z, z')}{k}$$

remains finite and approaches a limit as h and k approach 0, — then $f(z, z')$ is said to be an analytic function of the two independent variables z and z'. Since the publication of the book the investigations of Hartogs in the 62d volume of the *Mathematische Annalen* have shown that the condition of continuity is superfluous, being a consequence of the other conditions. Abel's theorem in the theory of the abelian integrals and Darboux's extension of Lagrange's series find a place in the latter chapter.

Passing now to the subject of differential equations, we find in Chapter XVIII an account of the elementary methods for solving total differential equations, or of reducing such an equation in certain cases to equations of lower order. Some properties of Riccati's equation are discussed in § 369, and in § 377 Euler's equation for the elliptic integrals is treated by means of Abel's theorem.

Chapter XIX, on the existence theorems, is exceedingly well done. First comes the proof of the existence of a solution of a system of ordinary differential equations of the first order and of a similar system of partial differential equations, the method being that of power series and *fonctions majorantes*. The proof in the latter case is especially simple and well arranged. Thus in three paragraphs, §§ 383, 384, and 386, all about existence theorems is contained which the reader needs for the cases of chief interest in what follows. But Cauchy's proof by means of a broken line, and the method of successive approximations are given in full. In § 387 the development of the fundamental properties of the solution of a system of ordinary differential equations begins, — properties which in the older treatments like those of Boole and Forsyth (1885) were assumed without proof and which obviously do not hold without restriction, even in simple cases. For example, according to those views, every differential equation of the first order,

$$\frac{dy}{dx}=f(x, y),$$

admits a solution containing an arbitrary constant,

$$y=\phi(x, C),$$

and the particular solution which assumes the value β when $x=\alpha$ can be obtained by solving the equation

$$\beta=\phi(\alpha, C)$$

for C. Furthermore, it was implied or explicitly stated in those earlier days that any function

$$y=\phi(x, C)$$

which is such that the elimination of C between the two equations

$$y=\phi(x, C), \qquad \frac{dy}{dx}=\frac{\partial\phi(x, C)}{\partial x}$$

leads to the given differential equation is *the* solution, *i. e.*, a solution with the above properties. Now, entirely aside from any questions of singular solutions or peculiar functions, this latter theorem is not true. Consider, for example, the differential equation

$$\frac{dy}{dx} = y.$$

A solution of this equation of the above kind is the following:

$$y = \phi(x, C) = e^{x+C}.$$

Moreover, the equation has a solution which vanishes when $x = a$, namely,

$$y = 0.$$

But the constant C cannot be so determined as to yield this solution.

In the early days of modern analysis crinkly curves and uncanny sets of points were most in evidence. But the real contribution of Cauchy and Weierstrass was to furnish the means of obtaining theorems at the threshold of a great subject like differential equations where no theorems of general validity had hitherto existed and where, therefore, no theorem hitherto known could be applied with security in any special case. The French early perceived this phase of progress in analysis, and the book before us is written in this spirit.

I know of no subject, unless it be the study of polynomials and elimination in algebra, that is better suited to give to the student who has just covered the elements of the theory of functions of one or more complex variables, power to apply the theory than a careful working out of the elements of the theory of differential equations, such as is outlined in the chapters before us. I use the word *outlined* advisedly. For while Professor Goursat sets forth in bold relief the main theorems of the theory, he does not in all cases develop the details to the point that the beginner must reach. This is a strong feature of the book for purposes of instruction. For example, the theorems about implicit functions have been carefully developed in their proper place, and they are frequently referred to in these chapters. But it happens time and again that the ultimate formulation is left to the student. And it is for this reason that I included those parts of algebra that deal with linear dependence and elimination, for it is here, where the attention is not distracted by difficulties in the theory of functions, that the student best learns what the problems of elimination really are.*

But I should give a wrong impression of Professor Goursat's work if I should fail to emphasize the wealth of illustration and the skill with which the leading ideas of a whole theory are at times brought out. Take, for example, this same chapter with the arid title: "Existence theorems." Here we find in three paragraphs at the end, entitled respectively: "One-parameter groups," "Applications to differential equations," and "Infinitesimal transformations," the greater part of the fundamental theorems and points of view of the first two thirds of Lie-Scheffer's book on Differentialgleichungen mit bekannten Infinitesimaltransformationen. Moreover, the theorems are here proven or so far formulated that the student can touch bottom. At least the latter statement becomes true if a slight alteration is made, being in substance what is done at the beginning of Lie-Engel, Transformationsgruppen; for thereby some of the difficulties that present themselves at the outset in connection with the conception of a group are satisfactorily met. The suggestion is this, that we restrict ourselves to the neighborhood of one of the points that is to be transformed and to that of the parametric values corresponding to the identical transformation. Thus for the one-parameter group in the plane we should require that the two functions

$$x' = f(x, y, a) \qquad y' = \phi(x, y, a)$$

satisfy the following conditions:

a) They shall both be analytic at a point (x_0, y_0, a_0);

b) If

$$x' = f(x, y, a), \qquad y' = \phi(x, y, a),$$

and

$$x'' = f(x', y', b), \qquad y'' = \phi(x', y', b),$$

then

$$x'' = f(x, y, c), \qquad y'' = \phi(x, y, c),$$

where (x, y) on the one hand, and a and b on the other, are chosen arbitrarily in the neighborhoods of the point (x_0, y_0) and the point a_0, and where, furthermore,

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* On this subject see, in particular, Bôcher, Introduction to Higher Algebra, Chaps. III, IV.

$$c = \psi(a, b)$$

is a function analytic in the point $a = a_0$, $b = b_0$;

c) $f(x, y, a_0) = x, \qquad \phi(x, y, a_0) = y,$

where (x, y) is an arbitrary point of the neighborhood of (x_0, y_0);

d) At least one of the derivatives $\partial f/\partial a$, $\partial\phi/\partial a$ shall not vanish at the point (x_0, y_0, a_0).

From b) and d) it follows that

$$\psi(a, a_0) = a, \qquad \psi(a_0, b) = b,$$

as also, in particular, that $\psi(a_0, a_0) = a_0$. And from c) we infer that the jacobian of the transformation,

$$\begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} \end{vmatrix},$$

does not vanish at the point (x_0, y_0, a_0), its value there being unity.

With these restrictions, all the theorems of the text are readily proven *im Kleinen* for the neighborhood of the point (x_0, y_0, a_0). By means of analytic continuation the theorems can then be extended, and thus the difficulties that present themselves at the beginning are reduced to a minimum.*

Chapter XX is devoted to linear differential equations, the equations of classical type being integrated explicitly and the developments of Fuchs in the neighborhood of a regular singular point being deduced. In Chapter XXI non-linear differential equations are studied. This chapter contains a good treatment of singular solutions.

But the author goes too far at times in omitting to state the restrictions which it is intended to impose. For example, consider § 427, which deals with the author's special investigations on congruences of curves. In order to avoid unnecessary generalities he very properly restricts himself to the case of two polynomials

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*The procedure in the first chapter of Lie-Engel, Transformationsgruppen, is similar, except that there sufficient conditions to insure the existence of the identical transformation and of inverse transformations are not introduced at the outset, this fact being explicitly stated. On the other hand, Lie-Engel are especially careful to call attention to the necessity of employing a sub-neighborhood, $((x))$, $((a))$ in their notation. The omission of this point by Professor Goursat does not contribute to accuracy or clearness.

$$F(x, y, z, a, b) = 0, \qquad \Phi(x, y, z, a, b) = 0.$$

But his arguments do not hold for polynomials, as is seen from the example

$$F(x, y, z, a, b) = (y - a)^2 - bx^2,$$
$$\Phi(x, y, z, a, b) = z - b,$$

both of these polynomials being, moreover, irreducible in the five arguments. It is not obvious what restrictions he means to observe. The above polynomials correspond to the system of differential equations

$$\frac{dy}{dx} = \sqrt{z}, \qquad \frac{dz}{dx} = 0.$$

Chapter XXII contains in sixty-three pages an admirable introduction to the elementary theory of partial differential equations. It is true that the proofs are not always carried down to such a point that the beginner will see how to make contact with his theorems about implicit functions; but he must be able to make such formulations for himself if he expects to do anything with modern analysis, and here is a good chance for him to learn his trade by working through in detail what is well defined in the text. Differential equations and simultaneous systems of the first order in three variables are studied and the theory is illustrated geometrically. Cauchy's problem and the method of characteristics are discussed, and the chapter closes with a paragraph on the Monge-Ampère equations of the second order.

The last chapter of the book, on the calculus of variations, gives an account of those problems and methods, with the development of which the school of Weierstrass has occupied itself, strong and weak variations, sufficient conditions, the E-function, fields of extremals, and Hilbert's invariant integral being the catch-words that suggest the developments.

In these two volumes Professor Goursat has given us a standard work on analysis, one that is modern in all respects, and which will undoubtedly be fully appreciated by students of mathematics and physics.

WILLIAM F. OSGOOD.

SHORTER NOTICES.

Calcul Graphique et Nomographie. Par M. D'OCAGNE. Paris, Octave Doin, 1908. xxvi + 381 pp.

Le Calcul Simplifié. Par M. D'OCAGNE. Paris, 1905. 225 pp.

THE first mentioned volume occurs in the division Sciences Appliquées of the Encyclopédie Scientifique, now publishing under the direction of Dr. Toulouse. It forms a part of the section Science du Calcul. The author's object is to develop as an independent discipline the theory of graphical processes of calculation. The work is therefore mainly to give system and revision to previously published material of his own and that of Culmann, Massau, Mehmke, and others. To this end the volume is divided into Livres I and II.

Livre I, Calcul Graphique, is subdivided into two chapters: Arithmétique et Algèbre graphique and Intégration graphique. In reading these two chapters one would perhaps better forget that any but graphical methods are available. Such methods of necessity often involve trying detail and sometimes sag under their own weight. They lead in the last step to the measurement of a segment by a scale. The author begins therefore with a discussion of "Echelles métriques," and easily converts one to the elegance of the "contre-échelle."

The dominating construction of this first book is that for graphic multiplication: "Si, entre deux parallèles, mM et nN à Oy (* * * lignes de rappel) separées par l'intervalle $mm = a$, nous tirons une droite MN de coefficient angulaire b, la différence NH ou c des ordonnées des points M et N est égal à ab. Pour avoir * * * b il suffit * * * ayant pris sur la partie negative de Ox le segment OQ égal à l'unité * * * de porter sur Oy le segment OQ égal à b, pris avec son signe, et de tirer PQ. La droite MN est dès lors parallèle à PQ." This paragraph is typical of the character and detail of presentation of much of the text.

The value of $a_1b_1 + a_2b_2 + \cdots + a_nb_n$ is then constructed by laying off the coefficients a on OX and b as angular coefficients consecutively and then measuring the segment on the nth *ligne de rappel* or ordinate. Such a construction serves also to solve the linear equation

$$a_0 + a_1 z_1 + \cdots + a_{n-1}z_{n-1} + a_n z_n = 0$$

in ∞^{n-1} ways by merely closing the polygon of angular coefficients on OX at a_n.

Again, a system of n linear equations of n unknowns can be solved directly if they form a "système étagé."

$$\begin{gathered}a_0 + a_1z_1 = 0,\\ b_0 + b_1z_1 + b_2z_2 = 0,\\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\ l_0 + l_1z + l_2z_2 + \cdots + l_nz_n = 0.\end{gathered}$$

By a construction due to Van den Berg any linear system may be reduced to the above form by graphic elimination. An alternative method for the general case is also presented.

A graphic solution of the general equation

$$a_0z^n + a_1z^{n-1} + a_2z^{n-2} + \cdots + a_n = 0$$

is based on the method of Lill, in which the coefficients are laid off in succession at right angles and the polygon of multiplication closed by trial and error.

A graphic method of multiplying $F(x)$ by x enables one to plot polynomials in x entirely by rule and compass; and graphic interpolation or determination of the coefficients of the parabola Π_n of the n^{th} degree

$$y = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n,$$

so that the graph passes through a large number of plotted points, is skillfully accomplished.

The second chapter on graphic integration aims at the evaluation of

$$\int F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) dx.$$

The reader must be content, however, if the shot is made at very close range and with a firm rest. The graph C with ordinate y is presupposed and is usually a Π_n, also it does not appear advisable to have F much worse than x^ny. The integral curve, whose ordinate measures the desired integral, is usually constructed by inscribing or circumscribing approximate polygons. When a Π_n is to be handled, the construction of the simple successive integrals can be made rigorous. One would scarcely resort, however, to graphic methods in such an emergency.

There is included in this chapter a section on the graphic integration of

$$dy/dx = f(x, y)$$

by approximate graph of the integral curves. The construction is dependent upon plotting the "isoclines"

$$f(x, y) = k.$$

Where possible, use may be made of the loci of inflections or cusps and the envelope.

The author has brought order and unity to a large variety of constructions and made available many scattered results. If the reader is disappointed at the calibre of some of the results, it is a solace to remember that they are *strictly* graphical throughout.

Livre II, Nomographie, takes one into a new atmosphere and presents in slightly over two hundred pages the essentials of this recent branch of mathematics. Those who have read the author's "Traité de Nomographie" with a desire for more compact treatment will find it here in excellent form. There is also a clear separation of the subject into its three essential parts: Représentation nomographique par lignes concourantes, * * * par points alignés, and * * * au moyen de points cotés diversement associées."

To give an idea of nomography itself the author's preface is illuminating: "Il est aisé de faire saisir a priori ce que constitue son essence au moyen d'une comparison familière. Toute le monde connait les graphiques au moyen desquels le Bureau central météorologique indique la repartition des hauteurs barométriques à une date donnée. Sur ces cartes figurent trois systèmes de lignes munis chacun d'une certaine chiffraison: les méridiens dont la cote est la longitude, les parallèles dont la cote est la latitude, les lignes, dites isobares, qui unissent tous les points où la hauteur barométrique atteint une même valeur, inscrite comme cote, à côté de chacune d'elles. Voici donc un nombre, la hauteur barométrique, que depend de deux autres, la longitude et la latitude, qui en est fonction, comme disent les mathematiciens, (* * *) *et dont on a la valeur en lisent simplement la cote d'une certaine ligne, l'isobare, passant par le point de rencontre de deux autres lignes,* le méridien et le parallèle, cotées au moyen des valeurs des nombres donnés."

Having set forth the notion of "echelles fonctionelles," viz., scales graduated with lengths proportional to $f(z)$ and inscribed with corresponding z, we are introduced to the important nomogram for

$$(1) \qquad f_1(z_1)g_3(z_3)+f_2(z_2)h_3(z_3)+f_3(z_3)=0,$$

which is solved once for all as follows:

Set $\qquad x=f_1(z_1), \quad y=f_2(z_2).$

Then (1) becomes

$$g_3x+h_3y+f_3=0,$$

a system of straight lines z_3. The solution for, say z_2, when z_1 and z_3 are known, requires us to read the z_2 on the line passing through the point of intersection of the lines marked z_1 and z_3. The idea of the function scales on the axes is due to Lalanne and was given the name *anamorphose.*

From the fact that the vanishing of a third order determinant determines the concurrence of three straight lines developes the most general equation in three variables

$$(2) \qquad \begin{vmatrix} f_1 & g_1 & h_1 \\ f_2 & g_2 & h_2 \\ f_3 & g_3 & h_3 \end{vmatrix} = 0,$$

whose nomogram has three systems of straight lines. This nomographic representation gives rise to a beautiful application of the projective transformation in the plane to obtain any desirable disposition of the straight lines involved.

It may be well to state here that the general problem, when and only when any given equation is representable by any given type or group of types of nomograms, is still in the main unsolved. Fortunately however nomography finds perhaps its greatest use in engineering where a formula can frequently be identified, by inspection, with some standard type. The question is of mathematical interest however and has been successfully treated in part by E. Duporcq and Goursat.

The most general nomogram of *lignes concourantes* which handles an equation of three variables will be one having three systems of inscribed curves. Furthermore if, instead of graduating the X and Y axes with function scales of one variable, we use the equations

$$x=f_{12}(z_1, z_2), \quad y=f_{34}(z_3, z_4),$$

we are in a fair way to realizing a nomogram for

$$F(f_{12}, f_{34}, z_5) = 0.$$

It remains only to adopt a wing nomogram on each axis to segregate their respective pairs of variables. The axes are then said to have each a *binary scale.*

It is to D'Ocagne himself, in 1884, that we owe "Représentation nomographique par points alignés." He points out clearly the origin in the idea of duality and still continues the constructions with line coordinates. The nomogram of three systems of straight lines gives place to one of three *curved scales*, the key to which is to read the third value where the line joining the given two cuts its scale. The line is of course a moveable straight edge. The advantage gained is appreciable with very little practice. For American engineers the idea of duality and use of parallel coordinates will be unattractive, but the results may be obtained entirely without the use of either as the author points out in Remarque II, page 229.

From this point of view the equation (2) is regarded as determining the alignment of the points defined by

$$x = \frac{f_i}{h_i}, \quad y = \frac{g_i}{h_i} \qquad (i = 1, 2, 3).$$

Such equations are classified by introducing the notion of *nomographic order* on which depends the number of curved and straight function scales. This idea of order is not in complete accord with that used by Soreau * and neither definition appears to determine the order of a given equation absolutely. There are other slight but unfortunate discrepancies in the notations of the two authors, for example, in the meaning of the term "*double alignement.*"

A more complete discussion would require us to reproduce figures and extend this review unduly. The book is thoroughly readable and emphasizes fewer unessential features than the larger treatise. A larger number of examples would improve it; for instance in connection with the nomograms of two straight scales and a web of curves. The volume closes with a note on the general theory.

The second volume appearing in the title is a compact treatment with many cuts and pictures of the following six topics: 1) Les instruments arithmétiques; 2) Les machines arithmétiques; 3) Les instruments et machines logarithmiques: 4) Les tables numériques (barèmes); 5) Les tracés graphiques; 6) Les tables graphiques (nomogrammes ou abaques). The volume also includes an interesting introduction and many historical notes.

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* R. Soreau, "Contribution à la théorie et aux applications de la Nomographie," *Mém. et Comptes rendus de la Soc. des Ing. civils*, 1901 (tiré à part).

L. I. HEWES.

Theorie der elliptischen Funktionen. Von H. DURÈGE. In fünfter Auflage neu bearbeitet von LUDWIG MAURER. Leipzig, B. G. Teubner, 1908. 436 + viii pp.

IT is difficult to understand on what ground this work can appropriately be called a new edition of Durège's well-known and admired book, since scarcely a trace of the original seems to have been left.

The first edition appeared in 1861, and was followed by successive editions at intervals of ten years or less, until the fourth in 1887, which was hardly more than a reprint of the third. There can be little doubt as to the inadvisability of further revising this work, now over twenty years old in its latest form, and which represents the state of the theory substantially as it was at the time of Jacobi. Since then this field has been transformed by the new theories of Weierstrass and Riemann, and has been more or less modified or influenced by other lines of mathematical activity such as the theory of groups and the Galois theory of equations. Moreover, the recently developed elliptic modular functions and the still more general class of the automorphic functions afford an extension or generalization which has not only placed the elliptic functions themselves in a new light, but has laid stress on their properties when the periods are regarded as additional independent variables.

It would evidently be quite out of the question to engraft all of these new methods and ideas on to the older theory as expounded by Durège, and Professor Maurer has started *de novo* in his treatment without attempting, as far as we can observe, to incorporate any of the material of the older work except that of course the jacobian functions are given their proper share of attention. The Weierstrassian functions and methods, however, predominate, and the influence of the great work of Klein and Fricke on the elliptic modular functions is observable throughout.

The first chapter treats of the elementary theory of the elliptic integrals, their transformation and reduction to normal forms. The second chapter considers the elliptic Riemann surface and the rational functions on it, while their integrals are discussed in the third chapter. The elliptic functions are studied in the next, and their analytic expressions in the fifth chapter. The sixth is devoted to such applications as the rectification of curves, parametric study of cubic curves, elliptic coordinates, the spherical pendulum, and the rotation of a body about its center of gravity. The seventh chapter is a brief resumé of the elements of group theory and the Galois theory of equations; the eighth treats of the division and transformation of the elliptic functions, and the ninth and last chapter deals with the modular functions. The important formulas are collected together at the end for convenience of reference, and a brief index closes the work.

We believe that Professor Maurer's book will prove to be one of the most useful that has been written on this subject. It embraces in a moderate compass a remarkably well considered and comprehensive view of a large and many-sided field.

J. I. HUTCHINSON.

Introduction to Infinitesimal Analysis. Functions of one Real Variable. By OSWALD VEBLEN and N. J. LENNES. New York, Wiley and Sons, 1907.

JUST fifteen years ago the first treatise in the English language on the theory of functions was published. Until then, this theory, which had been cultivated with great assiduity and success for half a century on the continent, was all but ignored by the English-speaking race, and in this country perhaps not half a dozen universities gave instruction in it. What a change has taken place in the few short years which have meanwhile elapsed! To-day all our better universities regularly give courses in this subject and, strange to relate, the most extended and scholarly treatise on the theory of functions of a complex variable is the work of an American.

During these years, a sister theory has come into prominence; we have now not only a theory of functions of complex variables, but also one for real variables. The work under review treats of this younger theory. Its two hundred and twenty odd pages contain a comprehensive and scholarly presentation of the foundation of the calculus, which on all sides is marked by originality of treatment and entire mastery of this intricate and difficult subject. The aim of the authors has been to write a book of moderate size which may be used as the basis of a short course or as a reference book in connection with a lecture course on this subject. The book has many praiseworthy features. Besides being very compact, the reasoning is entirely rigorous and correct, a point which, in view of the lax methods in vogue, cannot be too highly commended. The Borel-Lebesgue theorem on point sets included within intervals or rectangles is employed to establish uniform continuity and similar properties. We note as worthy of mention also the following features : the simple manner of introducing irrational numbers with the corresponding geometrical theory of points on the right line; the methodical use of the upper and lower bounds of a function for simplifying proofs, an extended comparison of functions with regard to magnitude (order, rank) with application to indeterminate forms and very general criteria for the convergence of improper integrals, finally the excellent treatment of integration which develops a surprising amount of the theory in a very small compass.

One who has carefully read this little volume will have acquired not only a goodly supply of facts, but, what is of far greater importance, familiarity with the exact and rigorous reasoning which is a sine qua non of further work in higher analysis.

JAMES PIERPONT.

An Algebra for Secondary Schools. By E. R. HEDRICK. American Book Company, 1908. x + 421 pp.

IT is with pleasure that we welcome this little volume into the already large family of elementary algebras. In these days of rapid changes, when so much that is time honored must make way for the new, there is always room for one more algebra, provided it is written like Professor Hedrick's in a thoroughly modern spirit which seeks to readjust both the material chosen and the manner of its presentation so as to satisfy more nearly the needs of the present day.

The book has many features well calculated to win support among progressive teachers. The style is clear and attractive. The genial personality of the author cannot be swallowed up even in a high school algebra. It pervades the whole book and makes it bright and fresh. Without departing too far from established usage, he has succeeded in giving several subjects a new aspect. As far as possible dead wood has been cut out and live material put in its place. The large number of new and attractive problems drawn from everyday life and from the elements of mensuration and physics will be greatly appreciated by the teacher. Considerable prominence has been given to graphical methods. The author has used them very effectively in treating proportion and variation, in the solution of equations, in establishing the notion of roots and logarithms, and even in explaining interpolation in a logarithmic table. Graphical methods not only make these subjects clearer and more real to the pupil, but will prove of great service to him whether he stops study with the high school or not. In the appendix some of the more advanced topics are treated as permutations and combinations, the binomial theorem, limits and infinite series, irrational and complex numbers.

It needs hardly to be added that the reasoning is sound and in accordance with present-day standards. The pupil will have no vicious habits of thought to eradicate if he goes on with his mathematics. In this connection we may note the emphasis the author lays on checking the solution of equations, a point too often entirely overlooked.

It is a good sign for the future of mathematics in this country when men of the calibre of Professor Hedrick are willing to take time from their own researches to write a work of this character. We wish his little book all success.

James Pierpont.

Die Lehre von den geometrischen Verwandtschaften. Erster Band: die Verwandtschaften zwischen Gebilden erster Stufe. By Rudolf Sturm. Leipzig and Berlin, B. G. Teubner, 1908. xii + 415 pp.

The treatise on geometric transformations, of which the present volume is the first part, is estimated to consist of four volumes, the manuscript of which is all in the hands of the printer. The others will be published as soon as possible.

While the first volume is concerned with forms of one dimension, it is not simply a text-book on projective geometry, but considers multiple correspondences, general involutions, trilinearity, and the problem of homography in two and in three dimensions. The first half of the book is concerned with parametric representation, anharmonic ratios, perspectivity, linear substitution, duality, involution, and polarity. Much stress is laid on directed elements, and all parametric representations are founded on the anharmonic property. The general method is synthetic, but so much use is made of algebraic substitutions that a long article (seventeen pages) is included, explaining the elements of the theory of invariants. All the steps of every proof are given, and a careful interpretation of each step is made. Indeed it would be easy for a patient student to acquire an accurate and extensive knowledge of many of the chapters of projective geometry by reading this book without a teacher. While the use of anharmonic ratios is by no means so great as was predicted for them a few decades ago, yet in parametric representation, theory of invariants, and theory of linear groups they have a lasting value, and most of these important interpretations are found in this book.

The footnotes give a large number of references, but too frequently these mention only the older literature. The technical words used in the text have become in part obsolete, and frequently have definitions different from those used by other writers. Fortunately the book is provided with an extensive glossary which largely atones for the confusion arising from the use of the older words.

The second half begins with an algebraic treatment of multiple correspondence, similar to that in the first volume of the author's treatise on line geometry,* but much more extensive. Halphen's precise application of Carnot's theorem is made the basis of the derivation of Plücker's formulas, and of the proof of the invariance of the genus under birational transformations.

The chapter on the generation of multiple correspondence is rather too abstract. This most useful tool in the study of curves and ruled surfaces frequently offers great difficulty to the reader, and a systematic exposition of the theory should be made as concrete as possible.

The word involution is used to define the usual involution, as well as symmetric (n, n) correspondences which are not involutional. An instructive example of multiple coincidences is followed by the warning to look out for very serious trouble in more complicated cases.

The existence of various involutions and multiple correspondences on the conic is treated in a synthetic manner, with the occasional introduction of an equation, which makes the theorems much clearer. In this regard, most of the treatment is less formal but more easily understood than the corresponding part of Reye's treatise.

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* Die Gebilde ersten und zweiten Grades der Liniengeometrie in synthetischer Behandlung, Leipzig, Teubner, 1892 ; vol. 2, 1893 ; vol. 3, 1896.

Since the author holds strictly to the traditions of the Steiner school (though never advocated by Steiner), no figures appear in the book. In many cases the explanations are sufficiently clear without this help, but in discussions of pencils of collineations, products of involutions, and similar subjects, the text could be made much briefer by the addition of appropriate figures.

The fifteen pages devoted to the twisted cubic curve contain a large number of results, skilfully arranged. The chapter on trilinearity is much more systematic than the treatment given this important subject in other places. A number of equations are used to establish the existence of neutral pairs, singular elements, and principal points, but much of the discussion is synthetic. The theory of the cubic surface follows easily from the elementary principles of trilinearity.

Nearly a hundred pages are occupied with the so-called problem of projectivity, *i. e.*, given a series of arbitrary points A_k, and their associates B_k, for any point P to find the corresponding point Q such that the pencil QB_k is projective with PA_k. The problem has no meaning if the number of arbitrary points is less than four, and becomes impossible when it is greater than seven. In particular, the Cremona transformation of degree five having six distinct base points results from five points A_k. The corresponding problem in space includes the discussion of the tetrahedral complex, and some surfaces belonging to it.

Some birational transformations are found in space which are believed to be new.

The later volumes are to contain the theory of collineations in space of two and of three dimensions, including the singular or degraded forms, and the theory of Cremona transformations and multiple correspondences. VIRGIL SNYDER.

Beiträge zur Theorie der linearen Transformationen als Einleitung in die algebraische Invariantentheorie. By W. SCHEIBNER. Leipzig, B. G. Teubner, 1907. 250 pp.

THE contents of most of this book appeared in the *Sächsische Berichte* from 1903 to 1907. The author explicitly declares that no new results are obtained and no essentially new method developed, but hopes by the repeated and consistent use of one method to awaken further scientific interest. The subject is developed along lines not too difficult for less mature readers, yet the point of view is sufficiently broad to justify the labor and patience involved in working through the maze of long algebraic processes. No use is made of symbolic notation, except that occasionally the author introduces various abbreviations of his own. A more remarkable restriction is that to non-homogeneous forms. This requires frequent paraphrases and the introduction of numerous unsymmetric formulas that are harder to master than the rather simple idea of homogeneous forms.

The ideas of invariant, covariant, source, hessian, jacobian, polarization, and transvectant are clearly treated in the first thirty pages, thus rapidly introducing the reader to the essence of the subject. The application to cubic and quartic forms is similar to that found in the treatise of Burnside and Panton, though more extensive and more systematic. A short chapter discusses the reduction of elliptic differentials to the canonical form; it is more direct than the procedure frequently followed, but presupposes greater maturity on the part of the reader.

Chapter IV, of 42 pages, is devoted to the determination of the system of invariants and covariants of quintic and sextic binary forms. This is the most interesting chapter in the book, although the goal is not always held clearly in view. The various steps are clear and follow the scheme laid down in the preceding chapters quite closely. The proof of the finiteness of the complete system is not completed, except that it is shown that the number of forms for the quintic cannot exceed twenty-three, and these are shown to be distinct by calculating the sources.

At times a paucity of words and a generous amount of algebraic detail lessen the pleasure of reading, but the subject matter makes any other procedure difficult. Unfortunately no geometric interpretation of the various forms is given, and the entire development proceeds strictly along algebraic lines.

The general subject matter of the book is followed by four appendices which are either applications of the methods discussed, or are concerned with linear transformations of non-algebraic functions. The first, of 10 pages, gives the geometric interpretation of the bilinear transformation in the complex plane. It is a distinct digression from the rest of the book, and is quite elementary. It was introduced to make the later appendices intelligible, but the treatment in these applications is so condensed that it can be of little value to a reader not having much more familiarity with conformal representation than that provided in the first appendix.

The second, of 20 pages, on the Tschirnhausen transformation, includes a detailed treatment of the reduction of the quintic to the Bring-Jerrard normal form; otherwise it is rather similar to the corresponding portion of Weber's Algebra.

The third appendix of 20 pages considers the solution of the icosahedron equation. The first half gives a very rapid survey of the Schwarzian derivative, the hypergeometric series, and the expression of the constants of transformation by means of gamma functions. The subject proper of this appendix is the working out of the problem suggested in Klein's Ikosaeder, page 139, *i. e.*, to start with the general binary quintic and deductively obtain the solution in terms of modular functions. The discussion is followed by a numerical illustration which greatly adds to its clearness.

The last appendix, of 40 pages, is concerned with linear transformations of elliptic theta functions and modular functions. As in the two preceding appendices, the amount of presupposed knowledge is much greater than in the book proper. The treatment is entirely transcendental, and has nothing in common with the preceding portions of the work. Numerous references show the relations between the present development and the existing literature, but it is not clear why this subject should be treated in a book on algebraic invariants.

VIRGIL SNYDER.

Table de Caractéristiques relatives a la Base 2310 des Facteurs premiers d'un Nombre inférieure a 30,030. By ERNEST LEBON. Paris, Delalain Frères, 1906. 32 pp.

IN this pamphlet the author has published a table of "characteristics with respect to the base 2310" by means of which any number between 1 and 30,030 can be readily factored. The first twelve pages are devoted to an explanation of the simple theory upon which the usefulness of the table is based, and to a description of the devices by means of which the characteristics are calculated. The last twenty pages contain the table itself. In a recent number of the BULLETIN, Professor

Fite has translated another article by Lebon in which the computation and use of a larger table, with base 30,030 = $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ instead of 2310, are described.*

In the factor tables of Glaisher, the numbers are listed and opposite to each is given the smallest prime divisor of the number. Lebon's idea is to divide the numbers prime to a given base B, *e. g.*, $B = 30,030$, into classes of the form

$$KB + I \qquad (K = 0, 1, 2, \cdots)$$

according to the values of their residues I with respect to B. The possible prime divisors D less than B are listed, and opposite to each and under each I is placed the characteristic k, if there is any, which specifies the smallest number divisible by D of the class belonging to I. It can then be readily discovered whether or not any other given number in the I-class is divisible by D, since for such a number $K - k$ must be divisible by D.

In the article which is the subject of this review, the characteristics with respect to the base 2310, for prime divisors from 13 to $\sqrt{30,030}$ have been listed, so that the table can be applied to the factorization of numbers between 1 and 30,030. If the characteristics for prime divisors up to 2310 had been given, the table would have been applicable to numbers as large as $(2310)^2 = 5,336,100$. The table described in the article translated by Professor Fite could be used to factor numbers between 1 and $(30,030)^2 = 901,899,900$. The factor tables already published by others, according to Lebon's statement in his introduction, give the factors of numbers from 1 to 10,000,000. All except those by Glaisher for the fourth, fifth, and sixth millions are out of print.

G. A. BLISS.

Serret's Lehrbuch der Differential- und Integralrechnung. Dritte Auflage, neu bearbeitet von GEORG SCHEFFERS. Zwei Bände: I, xvi + 624 pp.; II, xiv + 585 pp. Leipzig, B. G. Teubner, 1907.

SERRET'S name, which is in the title of this well known work, bears about the same relation to the edition under review that Webster's name bears to the latest edition of Webster's dictionary.

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* "Theory and construction of tables for the rapid determination of the prime factors of a number," BULLETIN, vol. 13 (1906–7), p. 74. The original article appeared in the *Comptes Rendus*, vol. 151 (1905), p. 78.

The original "Cours de calcul différentiel et intégral," appeared in the seventies and Harnack's translation into German was published in 1884. In 1897–99 a revised edition of this translation by Bohlmann appeared. This second edition, while perhaps the best-known text-book on the calculus in Germany, suffered greatly from the many typographical and mathematical errors. The figures were poorly drawn, many of them being altogether wrong. An incomplete list of over two hundred of these errors was published in connection with the volume on differential equations which was the third volume of the work.

This third edition, by Georg Scheffers, is not only revised, but entirely rewritten and in part rearranged. All the figures have been drawn anew and many new figures added. The theorems have been stated more clearly and more use has been made of italics to make them stand out from the body of the text. Many more references to previous paragraphs have been inserted.

Some of the more important changes which have been made in this edition are the following. The articles on number have been remodeled according to Dedekind's theory and the proofs of the theorems on continuous functions and on the convergence of series have thus been given a real backbone. A new chapter on implicit functions, with a thorough discussion of the functional determinant and of the independence of functions and of equations has been inserted. The difficulties of maxima and minima of functions of several variables receive considerable detailed discussion and the limitations of the parameter method of discussing an extreme value are clearly shown. Many improvements and corrections have been made in the chapters dealing with the application of the calculus to geometry. For instance, the plus or minus sign in the expression for the curvature of a space curve has been removed and in all cases where the square root occurs, a discussion of its sign has been given. The chapter on the complex variable has been entirely rewritten and shortened, the subject matter being limited to just what is necessary for the beginner to read understandingly the remainder of the work and the volume on differential equations which is now in preparation. The rather weak proof of the fundamental theorem of integral calculus in the second edition has been revised and completed. A proof of the theorem, "If $f(x, a)$ is a continuous function of x and a, then the integral

$$F(z, a) = \int_{x_0}^{z} f(x, a)dx$$

is a continuous function of z and a," is added to the chapter on definite integrals.

Harnack's appendix on Fourier's series and Fourier's integral has been kept unchanged, but several pages of explanatory notes have been inserted in the form of an introduction to the appendix. The valuable notes and references to other works, given at the end of the volumes in the second edition, have been omitted in the third edition for the rather insufficient reason that they might discourage the student.

This calculus is a geometer's calculus. Over three hundred of the twelve hundred pages are devoted to applications to geometry. With the exception of the paragraphs on center of gravity, there are practically no references to mechanics or physics. All the problems given are worked out in detail. In fact, detail is one of the features of the work. The reviser rejoices in saying that as far as possible he has eliminated from the text such phrases as "the reader will easily see," "the proof is left to the student."

Profiting by the example of the second edition, this edition is quite free from typographical errors. The few found by the reviewer are unimportant. But there is one which may be misleading. The symbol Δ in the second volume is not from the same font of type as that in the first volume, and the difference in the two is sufficient to cause confusion to a beginner.

A detailed table of contents and a copious index make the work very valuable as a reference book, and Serret's Lehrbuch will no doubt continue to be one of the most used books on the shelves in the mathematical reading room at Göttingen.

A. R. CRATHORNE.

Vorlesungen über mathematische Statistik (*Die Lehre von den statistischen Masszahlen*). Von ERNST BLASCHKE, Professor an der Technischen Hochschule in Wien. Leipzig, Teubner, 1906. 263 pp.

THE present work has to do with the theory and application of statistical constants (statistische Masszahlen). It opens with views of different writers as to the field to be included under mathematical statistics. While it contains but little that is new in the line of theory, nearly every point is accompanied by an apt illustration drawn from actual statistics. This with the many references to the literature makes the book very useful. The probability of life is kept in the foreground throughout. The author discriminates between mathematical and statistical probability, and holds that the two have a priori nothing in common. He gives methods of testing the identity of mathematical and statistical probability, that is to say, whether deviations are such as should be expected in the taking of a random sample.

The author divides statistical constants (Masszahlen) into intensive and extensive. Statistical probability belongs to the former and the various kinds of averages to the latter. On page 74 is the statement that the mode (dichteste Werte) is far the most important of the extensive statistical constants. The justification of this statement would be of interest when we consider it in connection with the almost universal acceptance of the arithmetic mean as a statistical average.

The representation of mortality by the formulas of Moivre, Lambert, Wittstein, and Babbage are classed as empirical representations, while the formula of Makeham is classed as an analytic representation. The chapter on the adjustment of observations contains in a clear form the methods of moments and least squares for fitting curves to observations; and criteria for the critical examination of the best adjustment.

Taken as a whole, the book is a useful contribution to that portion of mathematical statistics which finds its application in the probability of life.

H. L. Rietz.

Elektrische Wellen-Telegraphie, vier Vorlesungen gehalten von J. A. Fleming. Autorisierte deutsche Ausgabe von E. Aschkinass. Leipzig, B. G. Teubner, 1906. 185 pp.

In 1903 Fleming delivered the Cantor lectures before the London Society of Arts. These lectures on wireless telegraphy were for a popular audience and necessarily were descriptive and general, not mathematical. The translation by Aschkinass enables German readers to follow the author in their own language — which is very important for the class of readers for whom the lectures are primarily intended. Perhaps the most noteworthy characteristic of the style which is adopted is the constant close touch with nature and the forcefulness of analogy between disturbances in the ether, with which most of Fleming's audience were probably but poorly acquainted, and the disturbance due to sound in air with which they were certainly much more familiar. This method of exposition is typically English; it is found constantly in the writings of Faraday, Kelvin, J. J. Thomson, and other British physicists; it is met but rarely in the writings of French and German investigators even when they give popular expositions of physical science. Although analogies and mechanical models may be carried to a point where they confuse instead of enlighten and in many cases serve but to disguise the vital truth at the bottom of phenomena, it is probable that a greater use of them would be not without advantage to readers other than English. The translation of Fleming's lectures is therefore undoubtedly of considerable value to German students entirely apart from the matter which the lectures contain.

The first lecture proceeds at once to the presentation of electric force, magnetic induction, electrons, and electromagnetic waves. The analogy between the vibration and pressure in a closed organ pipe and the electric flux and tension on the antennæ of a wireless transmitter is set forth with great force. The function of the condenser is explained and numerous cuts illustrating different fundamental parts of apparatus devised for transmitting electromagnetic waves are given. The second and third lectures deal respectively with the chief points of the transmitting and receiving apparatus. The text by exposition and the cuts by illustration give the reader many of the smaller details as well as the general principles of the apparatus used in wireless telegraphy. In the fourth lecture are discussed numerous matters of a practical bearing on the availability of wireless telegraphy for commercial purposes. To all who realize the difficulties of the purely theoretical investigations in the transmission and reception of electromagnetic waves and who are aware of the many points in which the results of practical experience have been forced to come to the aid of the theory, it will at once appear that Fleming's lectures can be of considerable value not only to general readers but even to those who are primarily interested in and tolerably familiar with the purely theoretical side of the subject.

E. B. WILSON.

CORRECTION.

In the report of the sectional meetings of the Fourth International Congress of Mathematicians, published in the October number of the BULLETIN, an error occurs in the abstract of Professor Dingeldey's paper (on page 20 of the report), the third and fourth lines of which should read "... to generate the circle, the equilateral hyperbola, and the parabola ..."

NOTES.

THE annual meeting of the AMERICAN MATHEMATICAL SOCIETY will be held this year at Baltimore, on Wednesday and Thursday, December 30–31, in affiliation with the American association for the advancement of science. President H. S. WHITE's address, on "Bezout's theory of resultants and its influence on geometry," will be delivered at the opening of the afternoon session on Wednesday. The annual election will close on Thursday morning. Abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary as early as December 12.

THE twenty-third regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago, on Friday and Saturday, January 1–2, 1909. Titles and abstracts of papers intended for presentation at this meeting should be in the hands of the Secretary of the Section, Professor H. E. SLAUGHT, University of Chicago, on or before December 10.

THE American federation of teachers of science and mathematics will hold its annual meeting at Baltimore on Monday and Tuesday, December 28–29. Further details may be had of Professor H. W. TYLER, chairman of the executive committee, Massachusetts Institute of Technology, or of Professor C. R. MANN, Secretary, University of Chicago.

THE opening (October) number of volume 10 of the *Annals of Mathematics* contains the following papers: "On the small forced vibrations of systems with one degree of freedom," by MAXIME BÔCHER; "Two principles of map-making," by J.

K. WHITTEMORE; "Application of Tchebychef's principle in the projection of maps," by G. W. HILL; "The foundations of trigonometry," by A. C. LUNN; "Sufficient conditions for imaginary roots of algebraic equations," by OTTO DUNKEL.

AT the annual meeting of the London mathematical society, held November 12, the following papers were read: By W. BURNSIDE, "On the theory of groups of finite order" (presidential address); by J. E. LITTLEWOOD, "On the Dirichlet series and asymptotic expansion of integral functions of zero order"; by F. MORLEY, "The norm curves on a given base"; by J. O'SULLIVAN, "Satellite curves on a plane cubic"; by W. BURNSIDE, "On the arithmetic nature of the coefficients in a group of linear substitutions"; by E. W. HOBSON, "On the second mean-value theorem of the integral calculus," and "On the representation of a function by means of a series of Legendre's functions"; by H. BATEMAN, "The conformal representation of space of four dimensions with application to geometric optics"; by D. M. G. SOMMERVILLE, "Periodic properties of partitions"; by A. C. DIXON, "The solution of integral equations"; by A. L. DIXON, "The eliminant of three quantics in two independent variables"; by G. H. HARDY, "A note on the continuity or discontinuity of a function defined by an infinite product"; by F. B. PIDDUCK, "The energy and momentum of an ellipsoid electron"; by F. H. JACKSON, "On q-integration," and "On q-transformations of power series"; by T. STUART, "The complete solution in integers of the eulerian equation $x^4 + y^4 = u^4 + v^4$."

THE second annual meeting of the Italian association for the advancement of science was held at Florence, October 18 to 23, under the presidency of Professor V. VOLTERRA. Twenty sections participated at the meeting. In section I, mathematics, the following papers were presented: By L. AMOROSO, "On the extension of Dirichlet's principle to functions of several complex variables"; by T. BOGGIO, "Solution of some problems connected with the potential of a non-homogeneous sphere"; by U. CRUDELI, "Recent researches on the theory of figures of equilibrium of a liquid mass under uniform rotation"; by A. FAVARO, "Galileo's determination of the weight of the atmosphere"; by G. GIANFRANCESCHI, "Recent progress in electrodynamics of moving bodies"; by M. GREMIGNI, "Archimedes's postulate in the theory of geometric equivalence"; by G. Loria, "Geometrography and its transformations"; by P. Pizzetti, "Mathematics of geodesy"; by F. Severi, "Double integrals of the first kind belonging to an algebraic variety"; by E. Somigliana, "On the mechanical representation of some fields of force"; by A. Venturi, "On the theory of the balance of Eötrös"; by G. Vivanti, "On the present state of the theory of integral functions."

The next meeting of the association will be held at Padua, in September, 1909.

A general invitation has been sent to scientific men throughout the world to attend the meeting of the Australasian association for the advancement of science to be held at Brisbane, Queensland, January 11–17, 1909. Papers to be read before the association should be in the hands of the Secretary before December 26, 1908.

The firm of Martin Schilling in Halle announces the following new mathematical models: Planigraph, by Professor G. Koenigs, of the University of Paris; Thread model of the discriminant surface of the quintic equation in the normal form $u^5 + 10xu^3 + 5yu + z = 0$, by Dr. Mary E. Sinclair, of Oberlin College, under the direction of Professor O. Bolza, of the University of Chicago; The singularities of space curves, illustrated by nine models in pasteboard, by Miss Helga Lund, under the direction of Professor H. G. Zeuthen, of the University of Copenhagen. The first model is an apparatus for describing a plane by means of linkages; the second is a developable which divides space into five regions, according to the reality of the roots. Points on the cuspidal edge correspond to three equal roots. The third model includes the tangents and osculating planes to all the branches in each of the eight forms of singularities of space curves.

The following courses in mathematics are offered during the present winter semester. The courses in analytics, calculus, descriptive geometry and applied mathematics in the technical schools are obligatory.

University of Paris.—By Professor G. Darboux: Triply orthogonal systems, two hours.—By Professor E. Goursat: Theory of analytic functions, two hours.—By Professor P. PAINLEVÉ: Motion and equilibrium, two hours.—By Professor P. APPELL: General mathematics, two hours.—By Professor L. RAFFY: Theory of space curves (analytic), two hours.—By Professor H. POINCARÉ: Theory of the tides, two hours.—By Professor J. BOUSSINESQ: Thermodynamic properties of bodies, two hours.—By Professor E. BOREL: Theory of analytic continuation, two hours.

In the Ecole Normale.—By Professor J. TANNERY: Differential and integral calculus, two hours.—By Professor L. RAFFY: Applications of analysis to geometry, two hours.—By Professors E. BOREL and J. HADAMARD: General mathematics, two hours. Biweekly conferences of one hour each will be held by Professors RAFFY, HADAMARD, and P. PUISEUX and by Dr. E. BLUTEL.

BRAUNSCHWEIG.—By Professor R. DEDEKIND: Elements of the theory of numbers, three hours; Theory of probabilities, two hours.—By Professor R. FRICKE: Analytic geometry and algebra, five hours; Differential and integral calculus, five hours; Theory of functions, three hours.—By Professor W. LUDWIG: Descriptive geometry, five hours; Principles of higher mathematics, five hours.

CHARLOTTENBURG.—By Professors E. LAMPE, G. HETTNER and E. HAENTZSCHEL: Analytic geometry and the calculus, with exercises, six hours.—By Professors JOLLES, E. SALKOWSKI and G. SCHEFFERS: Descriptive geometry, eight hours.—By Professor E. STEINITZ: Algebra, four hours; Theory of potential, four hours; Calculus of variations, four hours (with Dr. C. CRANZ).—By Dr. R. FUCHS: Infinite series, two hours; Theory of functions, four hours.—By Professor O. KRIGAR-MENZEL: Theory of electricity, four hours; Theory of heat, four hours.—By Professor E. HAENTZSCHEL: Mechanics, four hours.

DARMSTADT.—By Professor F. DINGELDEY: Calculus, with exercises, six hours; Theory of Fourier series and integrals, three hours.—By Professor J. HORN: Calculus, with exercises, six hours.—By Professor H. WIENER: Descriptive geometry, with exercises, six hours.—By Professor R. MÜLLER: Descriptive geometry, with exercises, six hours; Analytic geometry, five hours.—By Professor L. HENNEBERG: Mechanics, six hours (two sections).—By Professor F. GRAEFE: History of mathematics, five hours; Review of elementary mathematics, with exercises, six hours.

MUNICH. — By Professor S. FINSTERWALDER: Analytic mechanics, five hours; Seminar, two hours. — By Professor W. v. DYCK: Differential equations, five hours; Seminar, one hour. — By Professor H. BURKHARDT: Mathematics in natural science, three hours. — By Professor L. BURMESTER: Descriptive geometry, with exercises, six hours (two sections). Sections in analytic geometry and the calculus will be conducted by Professors Finsterwalder, v. Dyck and Burkhardt, six hours. — By Dr. W. KUTTA: Review of algebra, six hours; Trigonometry, five hours; Seminar, two hours.

STUTTGART. — By Professor R. MEHMKE: Descriptive geometry, with exercises, six hours; Vector analysis, three hours; Seminar, two hours.—By Professor C. REUSCHLE: Analytic geometry of space, three hours; Selected chapters of plane analytic geometry and theory of invariants, four hours; Differential and integral calculus, with exercises, six hours; Seminar, two hours. — By Professor E. WÖLFFING: Differential and integral calculus, with exercises, six hours; Higher algebra, four hours. — By Dr. W. BRETSCHNEIDER: Review of elementary mathematics, four hours. — By Dr. E. STÜBLER: Elementary algebra and solution of numerical equations, four hours. — By : Trigonometry, four hours.

PROFESSOR CYPARISSOS STÉPHANOS, of the University of Athens, has been appointed rector of the university for the coming year.

PROFESSOR E. STUDY, of the University of Bonn, and Professor H. BURKHARDT, of the technical school at Munich, have declined the professorship of mathematics at the University of Leipzig.

PROFESSOR R. FRICKE, of the technical school at Brunswick, and Professor H. WIENER, of the technical school at Darmstadt, have been given the honorary title of Hofrat; Professor K. HENSEL, of the University of Marburg, has received the title of Geheimer Regierungsrat.

PROFESSOR E. NEUMANN, of the University of Marburg, has been promoted to a full professorship of mathematics.

Dr. C. Carathéodory, of the University of Bonn, has been promoted to an associate professorship of mathematics.

Dr. Esclangon, of the University of Bordeaux, has been promoted to an associate professorship of mathematics.

Professor R. C. Maclaurin, of Columbia University, has accepted the presidency of the Massachusetts Institute of Technology.

Professor C. A. Waldo, of Purdue University, has accepted the professorship of mathematics at Washington University, St. Louis. Professor A. M. Kenyon has been promoted to the head professorship of mathematics at Purdue.

Professor M. B. Porter, of the University of Texas, is spending a year abroad on leave of absence.

Professor R. G. D. Richardson, of Brown University, is spending a year's leave of absence in study at the University of Göttingen.

Dr. L. A. Howland has been appointed assistant professor of mathematics at Wesleyan University.

Dr. R. C. Archibald has been appointed professor of mathematics at Brown University.

At the University of Illinois Dr. A. R. Crathorne has been promoted to the position of associate in mathematics.

Mr. G. R. Clements, of Williams College, has been appointed instructor in mathematics at Harvard University.

Mr. S. A. Schwarz and Mr. W. A. Whyte have been appointed tutors in mathematics at the College of the City of New York.

Mr. A. R. Johnson has been appointed instructor in mathematics at Rutgers College.

Professor A. Korkin, of the University of St. Petersburg, died April 1, 1908, at the age of 71 years.

Recent catalogues of second-hand mathematical books: W. Junk, Berlin W. 15, Kurfürstendamm 201, bulletin no. 6, about 400 titles in mathematics. — Oswald Weigels Antiquarium, Leipzig, Königstrasse 1, catalogue no. 132, about 1500 titles in exact sciences.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Alteraugge (L.). Ueber lineare Relationen zwischen hypergeometrischen Integralen. (Diss.) Strassburg, 1907. 4to. 32 pp.

Askwith (E. H.). The analytic geometry of the conic sections. New York, Macmillan, 1908. 12mo. 14 + 442 pp. Cloth. $2.60

Boutroux (P.). Leçons sur les fonctions définies par les équations différentielles du premier ordre, professées au Collège de France. Avec une note de P. Painlevé. Paris, Gauthier-Villars, 1908. 8vo. 190 pp.

Breslau (H.). Dirichlets Satz von der arithmetischen Reihe für den Körper der dritten Einheitswurzeln. (Diss.) Strassburg, 1907. 8vo. 52 pp.

Cutroni (F.). Alcuni teoremi di geometria negli spazi a *n* dimensioni. Nota II: Gli angoli. Messina, D'Angelo, 1908. 8vo. 17 pp.

Dobrachowski (M.). Monographie der Kettenlinie. (Diss.) Bern, 1907. 8vo. 54 pp.

Fubini (G.). Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe. Pisa, Spoerri, 1908. 8vo. 13 + 416 pp. L. 15.00

Godfrey (C.) and Siddons (A. W.). Modern geometry. Cambridge, University Press, 1908. 8vo. 178 pp. Cloth. 4s. 6d.

Heywood (H. B.). Sur l'équation fonctionnelle de Fredholm et quelquesunes de ses applications. (Thèse.) Paris, Gauthier-Villars, 1908. 4to. 113 pp.

Hochheim (A.). Aufgaben aus der analytischen Geometrie der Ebene. 2es Heft: Die Kegelschnitte. Abteilung 1B: Auflösungen. 3te, vermehrte Auflage, bearbeitet von O. Jahn und F. Hochheim. Leipzig, Teubner, 1908. 8vo. 106 pp. M. 2.20

Hochheim (F.). See Hochheim (A.).

Huppery (R. W.). Analytische Untersuchung der allgemeinen Schraubenregelfläche. (Diss.) Strassburg, 1907. 8vo. 31 pp.

Jahn (O.). See Hochheim (A.).

Kowalewski (G.). See Newton.

Kummer (A.). Ueber eine Gattung von projektiven Transformationsgruppen in sechs Veränderlichen. (Diss.) Bonn, 1908. 8vo. 65 pp.

Laumann (T.). Ueber den Isomorphismus von Gruppen linearer Substitutionen mit reellen und mit komplexen Koeffizienten. (Diss.) Gottingen, 1907. 8vo. 63 pp.

Lindon (M.). Formeln aus der Differential- und Integralrechnung. Als Anhang zur Sohnckeschen Aufgabensammlung zusammengestellt. Jena, Schmidt, 1908. 8vo. 42 pp. M. 1.50

Loewenberg (G.). Was muss man von der analytischen Geometrie wissen? Berlin, Steinitz, 1908. 8vo. 80 pp. M. 1.00

Murray (D. A.). Differerential and integral calculus. New York, Longmans, 1908. 8vo. 18 + 491 pp. Cloth. $2.00

Newton's Abhandlung über die Quadratur der Kurven (1704). Aus dem Lateinischen übersetzt und herausgegeben von G. Kowalewski. (Ostwald's Klassiker der exakten Wissenschaften, 164.) Leipzig, Engelmann, 1908. 8vo. 66 pp. M. 1.50

Painlevé (P.). See Boutroux (P.).

Runge (C.). Analytische Geometrie der Ebene. Leipzig, Teubner, 1908. 8vo. 4 + 198 pp. Cloth. M. 6.00

Ruthinger (M.). Die Irreducibilitätsbeweise der Kreisteilungsgleichung. (Diss.) Strassburg, 1907. 8vo. 117 pp.

Schumacher (H. A.). Ueber eine Riemannsche Funktionenklasse mit zerfallender Thetafunktion. (Diss.) Strassburg, 1906. 8vo. 56 pp.

Siddons (A. W.). See Godfrey (C.).

Thomae (J.). Vorlesungen über bestimmte Integrale und die Fourierschen Reihen. Leipzig, Teubner, 1908. 8vo. 6 + 182 pp. Cloth. M. 7.80

Wright (J. E.). Invariants of quadratic differential forms. Cambridge, University Press, 1908. 8vo. 90 pp. 2s. 6d.

II. ELEMENTARY MATHEMATICS.

Amaldi (U.). See Enriques (F.).

Aubert (P.) et Papelier (G.). Exercices d'algèbre, d'analyse et de trigonométrie à l'usage des élèves de mathématiques spéciales. Paris, Vuibert, 1908. 8vo. 362 pp.

Ball (W. W. R.). Récréations mathématiques et problèmes des temps anciens et modernes. 2e édition française, d'après la 4e édition anglaise, et enrichie de nombreuses additions par J. Fitz-Patrick. 2e partie. Paris, Hermann, 1908. 16mo. 2 + 364 pp.

Bardey (E.). Algebraische Gleichungen nebst den Resultaten und den Methoden zu ihrer Auflösung. 6te Auflage. Bearbeitet von F. Pietzker. Leipzig, Teubner, 1908. 8vo. 12 + 420 pp. Cloth. M. 8.00

Barnard (S.) and Child (J. M.). A new algebra. Vol. I, containing parts 1, 2, 3, with answers. London, Macmillan, 1908. 8vo. 384 pp. Cloth. 2s. 6d.

Bergervoort (B. M.). Algebra-Regeln. Sexta bis Untersekunda einschliesslich. Bonn, Georgi, 1908. 8vo. 92 pp. M. 1.00

Bernstein (S. und F.). See Young (G. C. und W. H.).

Bersano (G. B.). Geometria, per la seconda e la terza classe delle scuole tecniche. Torino, Olivero, 1908. 8vo. 187 pp. L. 2.40

Bridgett (R. C.). Blackie's elementary modern algebra with answers. London, Blackie, 1908. 12mo. 232 pp. Cloth. 1s. 6d.

Busch (F.). See Féaux (B.).

Caldarera (F.). Primi fondamenti della geometria dello spazio seguito alla 1a parte, col titolo: *Primi fondamenti della geometria del piano.* Palermo, Virzi, 1908. 8vo. 4 + 235 pp. L. 7.00

Child (J. M.). See Barnard (S.).

Collins (J. V.). Practical elementary algebra. New York, American Book Co., 1908. 12mo. 8 + 420 pp. Half leather. $1.00

Combebrousse (C. de). Cours de mathématiques, à l'usage des candidats à l'Ecole polytechnique, à l'Ecole normale supérieure, à l'Ecole centrale des arts et manufactures. 5e édition, revue et augmentée. Vol. I, seconde partie : Algèbre élémentaire. Paris, Gauthier-Villars, 1907. 8vo. Pp. 295–796. Fr. 6.00

Cortázar (J.). Tratado de algebra elemental. 34a edición, corregida y arreglada. Madrid, 1908. 48 pp. P. 3.50

Davison (C.) and Richards (C. H.). Plane geometry for secondary schools. In four books. Cambridge, University Press, 1908. 8vo. Pp. 136, 85, 143, 81. Each, 1s.

Enriques (F.) e Amaldi (U.). Elementi di geometria, ad uso delle scuole secondarie superiori. 3a edizione. Bologna, Zanichelli, 1908. 16mo. 7 + 599 pp. L. 4.50

Féaux (B.). Buchstabenrechnung und Algebra, verbunden mit Aufgabensammlung. 11te, verbesserte und vermehrte Auflage, besorgt durch F. Busch. Paderborn, Schöningh, 1908. 8vo. 6 + 360 pp. M. 2.80

Ferval (H.). Eléments de trigonométrie. 4e édition, revue. Paris, Hachette, 1908. 16mo. 4 + 296 pp. Fr. 2.50

Fitz-Patrick (J.). See Ball (W. W. R.).

Grave (W.). Beiträge zur rationellen Didaktik mathematischer Disziplinen. (Diss.) Leipzig, 1908. 8vo. 88 pp.

Hedrick (E. R.). An algebra for secondary schools. New York, American Book Co., 1908. 12mo. 10 + 421 pp. Half leather. $1.00

Huntington (E. V.). Four-place tables of logarithms and trigonometric functions. Unabridged edition. Cambridge, Mass., Harvard Co-operative Society, 1908. 8vo. Cloth. $0.60

Intermediate applied mathematics papers; being the questions set at the University of London from 1882 to 1908. London, Clive, 1908. 8vo. 76 pp. 2s. 6d.

Lennes (N. J.). See Slaught (H. E.).

Lydon (N. S.). A preliminary geometry. London, Methuen, 1908. 8vo. 112 pp. 1s.

Lyman (E. A.). Solid geometry. New York, American Book Co., 1908. 12mo. 16 + 336 pp. Half leather. $0.75

Massari (V.). Elementi di algebra practica, ad uso delle scuole tecniche e normali. Roma, Voghera, 1908. 16mo. 4 + 128 pp. L. 1.50

Matriculation model answers : mathematics. Being the London University matriculation papers in mathematics from Sept., 1904 to June, 1908. London, Clive, 1908. 8vo. 178 pp. Cloth. 2s.

Metz (C.). Fünfstellige Logarithmen der Zahlen 1–10800 und der trigonometrischen Funktionen. Für den praktischen Gebrauch eingerichtet. Ausgabe A mit vollständigem Rand-Index. Berlin, Seydel, 1908. 8vo. 8 + 85 pp. Cloth. M. 4.00

Milne (W. J.). Standard algebra. New York, American Book Co., 1908. 12mo. 464 pp. Cloth. $1.00

NEVEU (H.). Cours d'algèbre à l'usage des classes de mathématiques élémentaires, des candidats à l'Ecole de Saint-Cyr et au professorat des Ecoles normales. 3e édition, revue et corrigée conformément aux derniers programmes officiels. Paris, Masson, 1908. 8vo. 671 pp.

NOCIONES elementales de geometría aplicadas al dibujo lineal, seguidas de las soluciones y respuestas, por los Hermanos de las Escuelas cristianas. Paris, 1908. 16mo. 31 pp.

PAPELIER (G.). See AUBERT (P.).

PIETZKER (F.). See BARDEY (E.).

RICHARDS (C. H.). See DAVISON (C.).

SACHS (J.). Tafeln zum mathematischen Unterricht. Leipzig, Teubner, 1908. 8vo. 2 + 120 pp. M. 6.00

SCOONES (P.) and TODD (L.). The Eton algebra, Part I, with answers. London, Macmillan, 1908. 8vo. 210 pp. Cloth. 2s. 6d.

SCOTTI (G.). Elementi di geometria, ad uso del ginnasio superiore. 4a edizione. Torino, Salesiana, 1908. 8vo. 127 pp.

SLAUGHT (H. E.) and LENNES (N. J.). High school algebra; complete course. Boston, Allyn & Bacon, 1908. 12mo. 15 + 494 pp. Cloth. $1.20

SMOLEY (C.). Five-place logarithmic-trigonometric tables. London, Constable, 1908. 8vo. 6s.

——. Parallel tables of logarithms and squares, etc. 5th edition, revised. London, Constable, 1908. 8vo. 14s.

SPIEKER (T.). Lehrbuch der ebenen Geometrie mit Uebungs-Aufgaben für höhere Lehranstalten. Ausgabe C. Abgekürzte Kurse. 4te Auflage. Potsdam, Stein, 1908. 8vo. 4 + 206 pp. M. 2.50

STÜCK (E.). Vierstellige logarithmische und trigonometrische Tafeln. Herausgegeben auf Veranlassung des nautischen Departements des Reichs-Marineamts. Wilhelmshaven, Lohse, 1908. 8vo. 4 + 94 pp. Cloth. M. 1.25

TODD (L.). See SCOONES (P.).

TOURS (F. I. C.). Tables de logarithmes à cinq décimales pour les nombres de 1 à 10,000, et pour les fonctions trigonométriques de minute en minute. Paris, Poussielgue, 1908. 16mo. 8 + 144 pp.

TOURS (F. J.). Eléments de trigonométrie rectiligne avec de nombreux exercices. Paris, Poussielgue, 1908. 16mo. 7 + 257 pp.

WOLFF (H.). Sätze und Aufgaben der Geometrie für Realanstalten, nach den preussischen Lehrplänen von 1901 zusammengestellt. 1er Teil: Pensum der Unterstufe (Quarta bis Untersekunda). Leipzig, Teubner, 1908. 8vo. 4 + 98 pp. M. 1.60

YOUNG (G. C. and W. H.). Der kleine Geometer, Deutsche Ausgabe, besorgt von S. und F. Bernstein. Leipzig, Teubner, 1908. 8vo. 16 + 239 pp. Cloth. M. 3.00

ZORETTI (L.). Leçons d'algèbre, classes de mathématiques A et B, préparation aux Ecoles navales et de Saint-Cyr. Paris, Paulin, 1908. 16mo. 3 + 461 pp. Fr. 6.00

III. APPLIED MATHEMATICS.

Ball (R. S.). Meccanica. Traduzione di J. Benetti. 4a edizione, riveduta. Milano, Hoepli, 1908. 16mo. 16 + 198 pp.

Bedford (E. J.). Introductory mechancis; with diagrams. New York, Longmans, 1908. 16mo. 10 + 141 pp. Cloth. $0.50

Benetti (J.). See Ball (R. S.).

Brillouin (M.). Mémoire sur l'ellipticité du géoïde dans le tunnel du Simplon. Paris, 1908. 8vo. 230 pp.

Calculations of elements of machine design. New York, Industrial Press, 1908. 8vo. 48 pp. $0.25

Church (I. P.). Mechanics of engineering, for use in technical schools. Revised edition, partly rewritten. New York, Wiley, 1908. 8vo. 26 + 854 pp. Cloth. $6.00

Cross (C. R.). Notes on mechanics, printed for the use of students in the Massachusetts Institute of Technology. Boston, Cross, 1908. 8vo. 75 pp. (Not for sale.)

Czuber (E.). Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik und Lebensversicherung. Vol. I. Wahrscheinlichkeitstheorie, Fehlerausgleichung, Kollektivmasslehre. 2te, sorgfältig durchgesehene und erweiterte Auflage. (Teubners Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Vol. IX, 1.) Leipzig, Teubner, 1908. 8vo. 10 + 410 pp. Cloth. M. 12.00.

Diamant (S.). Curves for calculating beams, channels and reactions. London, Spon, 1908. 8vo. Cloth. 8s. 6d.

Dobbs (W. J.). Examples in elementary mechanics: practical, graphical and theoretical. London, Methuen, 1908. 8vo. 356 pp. Cloth. 5s.

Donath (A.). Lehrbuch der Elektromechanik. Eine Darstellung der wissenschaftlichen Grundlagen der Elektrotechnik und ihrer praktischen Anwendung in ausführlicher, leichtverständlicher Form. Zum Gebrauche für technische Lehranstalten, sowie ganz besonders zum Selbstunterricht. Vol. I. Der Magnetismus, der Elektromagnetismus und die Berechnung der Feldmagnete von Dynamomaschinen und Elektromotoren. Mit zahlreichen Rechnungsbeispielen. 2ter Teil. Jena, Costenoble, 1908. 8vo. Pp. 15–18 and 385–731. M. 6.00

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LIBRARY.

THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and fortieth regular meeting of the Society was held in New York City on Saturday, October 31, 1908, extending through a single morning session. The attendance included the following twenty-one members:

Professor E. W. Brown, Professor F. N. Cole, Miss E. B. Cowley, Miss L. D. Cummings, Professor L. P. Eisenhart, Professor T. S. Fiske, Professor Edward Kasner, Mr. W. C. Krathwohl, Professor James Maclay, Mr. H. F. MacNeish, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Mr. L. P. Siceloff, Professor D. E. Smith, Dr. Elijah Swift, Professor H. D. Thompson, Professor J. M. Van Vleck, Professor Oswald Veblen, Professor H. S. White, Professor J. K. Whittemore.

The President of the Society, President H. S. White, occupied the chair. The Council announced the election of the following persons to membership in the Society: Professor J. A. Brewster, St. Angela's College; Professor W. H. Butts, University of Michigan; Dr. C. F. Craig, Cornell University; Professor T. A. Martin, Mt. Union College; Professor M. T. Peed, Emory College; Mr. G. E. Roosevelt, New York, N. Y.; Mr. L. M. Saxton, College of the City of New York. Four applications for admission to membership in the Society were received.

A list of nominations of officers and other members of the Council was adopted and ordered placed on the official ballot for the annual election at the December meeting.

It was decided to hold the annual meeting at Baltimore, on Wednesday and Thursday, December 30–31, in affiliation with the American association for the advancement of science.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On the theory of functions of a triple variable."

(2) Professor R. D. CARMICHAEL: "Notes on the simplex theory of numbers."

(3) Professor EDWARD KASNER: "Conformality and functions of two or more complex variables (second paper)."

(4) Professor G. A. MILLER: "On the groups generated by two operators satisfying the equation $s_1s_2 = s_2^{\alpha}s_1^{\beta}$."

(5) Professor E. B. WILSON: "The number of types of collineations."

(6) Dr. FRANK IRWIN: "The invariants of linear differential expressions."

(7) Dr. A. E. LANDRY: "A geometrical application of binary syzygies."

Dr. Landry's paper was communicated to the Society through Professor White, Dr. Irwin's paper through Professor Veblen. In the absence of the authors the papers of Professor Carmichael, Professor Miller, Professor Wilson, and Dr. Landry were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael defines a system of triple numbers (an extension of ordinary complex numbers) and introduces functions of a triple variable thus defined. For these numbers all the formal laws of operation of ordinary complex algebra are found to hold. It follows then from a well-known theorem that the zero product theorem cannot hold in its general form; but it turns out that only a slight modification of this theorem is necessary and that, with the small exceptions to which this gives rise, algebraic operations with these triple numbers may be performed as with ordinary complex numbers.

In particular it is found that the Cauchy and Laurent expansions hold practically without modification. It becomes then a simple matter to write out the properties of holomorphic functions of this triple variable through their analogies with those in the function theory of ordinary complex numbers.

Numbers and operations in this triple system have a geometrical interpretation in relation to space as those in the complex system have in relation to the plane.

2. Professor Carmichael's second paper consists of three notes:

I. The first deals with the factors of the continued product of an arithmetical series of integers. A formula is found for the highest power of a prime p contained as a factor in this product. Some simple forms for special cases are worked out.

II. This note deals with an extension of Fermat's theorem, and a new function $M(a)$ of an integer a is defined. Its values for every possible $a \leqq 250$ are tabulated.

III. This states briefly a method for finding all the solutions of $\phi(z) = a$, where $\phi(z)$ denotes the totient of a.

3. In a paper presented at the summer meeting of the Society Professor Kasner discussed the group of transformations defined by pairs of analytic functions $\phi(z, w)$, $\psi(z, w)$ of two independent complex variables $z = x + iy$, $w = x' + iy'$, which were regarded as representing a point in four-dimensional space. In the present paper the variables z, w are taken to represent an ordered point pair or vector in the plane. The vector transformations which then arise are characterized by this property: If any three neighboring vectors are so situated that the infinitesimal triangles formed by their initial and terminal points are similar, the same is true of the transformed vectors (the new triangles are then necessarily similar to the original triangles). Extensions to n variables and to conjugate variables are easily made.

The results are applied to Poincaré's generalization of the Dirichlet problem (*Palermo Rendiconti*, 1907). As a supplement to Poincaré's discussion of three-dimensional varieties, it is shown that not all two-dimensional varieties are equivalent. Finally the linear transformations which appear in Picard's theory of hyperfuchsian functions are shown to admit of simple interpretation.

4. It is first pointed out by Professor Miller that all the groups of genus zero belong in the category discussed. He then takes up the relation $s_1s_2 = s_2^2s_1^2$, first studied by Cayley in 1878 and more recently by Netto in 1905, and greatly extends their results. Among the new theorems proved are the following: If two operators of order 6 satisfy the condition $s_1s_2 = s_2^2s_1^2$, the order of s_1s_2 cannot be odd unless one of these operators is the inverse of the other. When s_1s_2 is of order 2, 4, or 6, the groups generated by s_1, s_2 have orders which divide 24, 192, or 648 respectively, and they may actually have these orders. If two operators of orders 4 and 6 respectively satisfy the condition $s_1s_2 = s_2^3s_1^2$, they generate a group of order 48 involving the direct product of the alternating group of order 12 and the group of order 2. The quotient group of this group of order 48 with respect to the subgroup generated by the cube of the operator of order 6 is the symmetric group of order 24. If two operators of orders 4 and 12 respectively satisfy the condition

$s_1s_2 = s_2^3s_1^2$, they generate a group of order 96 involving the quaternion group as a characteristic subgroup.

The most important results obtained relate to the case where the two generating operators of a group satisfy the relation $s_1s_2 = s_2^{-2}s_1^{-2}$. It is first proved that either two such operators must have the same order or the order of one is three times that of the other. Moreover, the order of s_1s_2 must be odd, and the following relations are always true $(s_1s_2)^3 = s_1^3s_2^3$, $(s_1s_2)^{11} = s_1^5s_2^5$. By means of these relations a number of special cases can readily be considered. Some of these special results are expressed in the following theorem: If two operators of order 6 satisfy the condition $s_1s_2 = s_2^4s_1^4$, they generate one of the following groups: the cyclic group of order 6, the direct product of the group of order 2 and the non-cyclic group of order 9, the metacyclic group of order 42, or a group of order 126 involving an invariant cyclic subgroup of order 21.

5. In a short note, which has since been published in the *Jahresbericht der deutschen Mathematiker-Vereinigung*, on the number of types of collineations, Professor Wilson tabulates the number of types of non-singular collineations, classified according to the different systems of elementary divisors, when the number of homogeneous variables is twelve or less. It is found that the number of types is nearly doubled on passing from one dimension to the next higher throughout the range of dimensions here considered.

6. Dr. Irwin's paper deals with linear differential expressions of the nth order, both ordinary and partial, and their invariants under (1) change of dependent variable, (2) multiplication by a function of the independent variables, (3) change of independent variables. Cockle, in the *Philosophical Magazine*, 1870, handled the problem, for an ordinary differential equation under change of dependent variable, $u = \psi \cdot \eta$ by reducing the equation to a "canonical" form in which the coefficient of the $(n-1)$st derivative vanishes. If A be any coefficient of this canonical form, then A/ψ is a rational invariant, as well as $\psi^{-1}d^lA/d\psi^l$, and any invariant can be expressed as a function of these invariants. The writer finds that this method may be applied, with parallel results throughout, to ordinary differential expressions under change of independent variable. But, further, the attempt to apply the method to partial dif-

ferential expressions under change of dependent variable leads to results. Here we get for ψ the equations $\partial \log \psi / \partial x_i = K_i$, the K's being functions of the coefficients of the given differential expression, equations that are solvable only if the invariants $\partial K_i / \partial x_j - \partial K_j / \partial x_i$ vanish. Nevertheless, if we calculate the higher derivatives of ψ from $\partial \psi / \partial x_i = K_i \psi$ *in any manner whatever* and substitute in any coefficient A of the transformed differential expression, A/ψ will be an invariant. We also have invariants analogous to the invariants $\psi^{-1} d^i A / d\psi^i$ of ordinary differential expressions.

Lastly, the possibility of making a differential expression of the second order self-adjoint by multiplying it by a function of the independent variables is found to depend on the identical vanishing of an expression covariant for all three transformations.

7. Given a curve $J^{(m+1)}$ of order $m+1$ and having an m-fold point, and a fundamental involution I_1^{2m-1} set up thereon, the problem dealt with in Dr. Landry's paper is how to determine other curves, of the same type and with the same multiple point, which intersect the given curve either at the base points of the I_1^{2m-1} or in sets of points given by covariants of the form which vanishes for the base points.

Let it be required to find the curve which intersects the $J^{(m+1)}$ at the vanishing points of a $C_{k,n}$. If we use homogeneous coordinates x_0, x_1, x_2 and place the multiple point at the vertex (1, 0, 0) of the reference triangle, the equation of the $J^{(m+1)}$ will be

$$x_0 C_{m,m}(x_1, x_2) = C_{p,m+1}(x_1, x_2),$$

where $C_{m,m}$ is the canonizant of the fundamental form $C_{1,2m-1}$.

The sought curve will have an equation of the form

$$x_0 C_{q,n-m-1} = C_{r,n-m};$$

and on eliminating x_0 we find that the intersections are given by

$$C_{p,m+1} C_{q,n-m-1} - C_{m,m} C_{r,n-m} = 0,$$

that is, we must have identically

$$C_{k,n} I_{p+q-k} \equiv C_{p,m+1} C_{q,n-m-1} - C_{m,m} C_{r,n-m};$$

so that the problem leads to a syzygy among the covariants of the fundamental form.

Next, the form of the invariant multiplier is investigated, and it is shown that the eliminant of $C_{m,n}$ and $C_{p,m+1}$ is a perfect mth power, of which I_{p+q-k} is the mth root. Lastly, the special cases of the quintic and septimic are discussed. The first case is treated exhaustively; for the second, on the other hand, only a summary of the main results obtained, without proofs, is given.

The paper will be published in the *Proceedings of the American Academy of Arts and Sciences.*

F. N. COLE,
Secretary.

ON THE GROUPS GENERATED BY TWO OPERATORS SATISFYING THE CONDITION $s_1s_2=s_2^{-2}s_1^{-2}$.

BY PROFESSOR G. A. MILLER.

(Read before the Southwestern Section of the American Mathematical Society, November 28, 1908).

§ 1. *Introduction.*

REMARKABLE general properties may easily be proved as regards the system of groups generated by two operators s_1, s_2 which satisfy the condition expressed by one of the following pair of equivalent equations:

$$s_1s_2=s_2^{-2}s_1^{-2},\quad s_1^2s_2^2=s_2^{-1}s_1^{-1}.$$

From the facts that $(s_2s_1)^2=s_2s_1s_2s_1=s_2^{-1}s_1^{-1}=(s_1s_2)^{-1}$ and that s_1s_2 is of the same order as s_2s_1, it results that the order of s_1s_2 is an odd number. From the same equations it follows that s_1s_2 is transformed into a power of itself both by s_1 and by s_2, and hence the cyclic group generated by s_1s_2 is invariant under the group G generated by s_1, s_2. As G is generated both by s_1s_2, s_1 and also by s_1s_2, s_2 it results from the preceding sentence that if s_1^{α} is the lowest power of s_1 which occurs in the cyclic group generated by s_1s_2 it is necessary that s_2^{α} is the lowest power of s_2 occurring in this group and vice versa. Moreover, if s_1 is commutative with $(s_1s_2)^n$ the following equations are satisfied:

$$(s_2s_1)^n=(s_1s_2)^n=(s_2s_1)^{-2n}\quad\text{since}\quad(s_2s_1)^2=s_2^{-1}s_1^{-1}=(s_1s_2)^{-1}.$$

As similar remarks evidently apply to s_2, we have proved the following theorem:

If two operators satisfy the condition $s_1s_2 = s_2^{-2}s_1^{-2}$, *either they are of the same order, or the order of one is three times that of the other.*

§ 2. *Powers of* s_2s_1 *and* s_1s_2.

The following equations may readily be verified:

$$(s_2s_1)^2 = s_2s_1s_2s_1 = s_2^{-1}s_1^{-1} = s_1^2s_2^2, \qquad (s_2s_1)^3 = s_1^2s_2^3s_1,$$

$$(s_2s_1)^4 = s_1^2s_2^3s_1s_2s_1 = s_1^2s_2s_1^{-1} = s_1^2\cdot s_2s_1\cdot s_1^{-2}.$$

From the last of these it follows that

$$(s_2s_1)^{4n} = s_1^2(s_2s_1)^ns_1^{-2}.$$

Combining this equation with $(s_2s_1)^2 = s_1^2s_2^2$, there results the formula $(s_2s_1)^{4n+2} = s_1^2(s_2s_1)^ns_2^2$. By giving n successively the previous values of $4n+2$, we arrive at the equations

$$(s_2s_1)^2 = s_1^2s_2^2, \quad (s_2s_1)^{10} = s_1^4s_2^4, \quad (s_2s_1)^{42} = s_1^6s_2^6, \quad (s_2s_1)^{170} = s_1^8s_2^8,$$

etc. In general, it results by induction that

$$(s_2s_1)^{\frac{2}{3}(4^m-1)} = s_1^{2m}s_2^{2m}. \tag{A}$$

Since $4^m \equiv 1 \bmod 3$, it follows that $\frac{2}{3}(4^m-1)$ is always an even integer, hence it must be divisible by double the order of s_2s_1 whenever the orders of s_1, s_2 divide $2m$. In particular, if the order of each of the operators s_1, s_2 is 4, G is the metacyclic group of order 20.*

In a similar way it is easy to find general formulas for powers of s_1s_2, as follows:

$$(s_1s_2)^3 = s_1s_2s_1s_2s_1s_2 = s_1s_2^{-1}s_1^{-1}s_2 = s_1^3s_2^3,$$

$$(s_1s_2)^4 = s_1^3s_2s_1^{-2} = s_1^2\cdot s_1s_2\cdot s_1^{-2}.$$

Hence

$$(s_1s_2)^{4n} = s_1^2(s_1s_2)^ns_1^{-2} \quad \text{and} \quad (s_1s_2)^{4n-1} = s_1^2(s_1s_2)^ns_2^2.$$

As before, we assign to n successively the values obtained from the form $4^n - 1$ and thus arrive at the equations

$$(s_1s_2)^3 = s_1^3s_2^3, \quad (s_1s_2)^{11} = s_1^5s_2^5, \quad (s_1s_2)^{43} = s_1^7s_2^7, \quad (s_1s_2)^{171} = s_1^9s_2^9,$$

etc. The general formula, which may easily be proved by induction, is

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* Cf. Netto, *Crelle*, vol. 128 (1905), p. 254.

$$(s_1s_2)^{\frac{2}{3}4^m+\frac{1}{3}} = s_1^{2m+1}s_2^{2m+1}. \qquad (B)$$

Since $\frac{2}{3}4^m+\frac{1}{3} = 4^m - \frac{1}{3}(4^m - 1) = 4^m -$ an odd integer, it is clear that the exponent s_1s_2 is odd for every value of m in this general formula. That is, formula A exhibits a relation between the products of even powers of s_1, s_2 and even powers of s_2s_1, while formula B is restricted to odd powers.

§ 3. *Conclusion.*

From the Introduction and formulas A, B it results that the order of G is a finite number whenever the orders of s_1, s_2 are finite; and that all the operators of G may be represented in the form $s_1^{\alpha}(s_1s_2)^{\beta}$. It is, however, not always possible to represent these operators in the form $s_1^{\alpha}s_2^{\beta}$, as is evident from the metacyclic group mentioned in § 2. Suppose that s_1, s_2 have the same order prime to 3, and let t be an operator of order 3 which is commutative with each of the operators s_1, s_2. As s_1, s_2 satisfy the condition $s_1s_2 = s_2^{-2}s_1^{-2}$, it is evident that s_1, $s_2t = t_1$ satisfy the condition $s_1t_1 = t_1^{-2}s_1^{-2}$ and that they generate a group whose order is three times that of G. This illustrates the fact that the order of one of the two operators s_1, s_2 may actually be three times that of the other, and hence the theorem at the end of § 1 relates to actual cases. As $(s_2s_1)^3 = s_2s_1s_2s_1s_2s_1 = s_2^{-1}s_1^{-1}s_2s_1$ and the cyclic group generated by s_2s_1 is invariant under G, it results that *the third power of every operator in the cyclic group generated by s_2s_1 is a commutator of G and all the commutators are such third powers.* Similar remarks clearly apply to s_1s_2, and this theorem could be deduced from the results proved in § 1.

It may be of interest to observe the analogy between the case under consideration and the relation $s_1s_2 = s_2^2s_1^2$, which was studied by Cayley * as early as 1878 and by others † more recently. Although this relation has received considerable attention yet it has given rise to only a few theorems of general interest and it presents many diffiulties which have not yet been overcome. On the other hand, the relation $s_1s_2 = s_2^{-2}s_1^{-2}$ gives rise to a number of general theorems and restricts G to a well-known category of groups. In each case the orders of s_1 and s_2 may be equal to each other. If they are unequal the order of one must be an odd number while that of the other is twice this odd number when $s_1s_2 = s_2^2s_1^2$; but when $s_1s_2 = s_2^{-2}s_1^{-2}$ it is only necessary that the order of one is three times that of the other.

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* *Messenger of Mathematics*, vol. 7 (1878), p. 188.

† Cf. Netto, *Crelle*, vol. 128 (1905), p. 251.

University of Illinois.

THE TEACHING OF MECHANICS.

An Elementary Treatise on Theoretical Mechanics. By J. H. Jeans. Ginn and Company, 1907. viii + 364 pp.

There are few topics in elementary mathematics that are more generally mishandled by the writers of our text books than Newton's three laws of motion. Perhaps it would be more accurate to say that the applications of the laws are generally misunderstood and that consequently the exponents of axioms which form the foundation of the mathematical science of mechanics rarely fail to make some fundamental error which destroys at the outset any hope of a logical development of the subject. The worst cases generally occur in the books which are published under the title "Physics." It frequently happens that the authors have not mastered the meaning of the laws; more frequently they show a want of care in their statements and explanations. In either case the effect on the student must be the same — a nebulous conception of the whole subject and a general impression that one can get along perfectly well in physics or engineering without bothering to understand what facts are directly observed and which can be deduced from the laws of motion. A man who wishes to rise to the higher levels of these professions must know such matters.

Perhaps it will not be altogether out of place to insist here on some points in the teaching in this country of applied mathematics or mathematical physics, whatever be the name we like to give to the science which concerns itself with the application of mathematics to problems in which space, time, and matter are supposed to be related by certain definitely stated laws. At the outset, the subject is a "pure" science in exactly the same way that pure mathematics is so, in that it rests solely on definitions and axioms which have no *necessary* relation to the phenomena of nature. Every problem attacked is an ideal, not an actual problem. The statement of the ideal problem must conform to the laws laid down if it is to fall within the province of the subject. In order that a real physical problem may be treated by the science, it is necessary that the laws of the science should be the same as the laws which govern the phenomena of nature. Here the "application" comes in. We frame the laws of the science in such a manner that we may imitate as far as possible the laws which are supposed to govern natural phenomena. In this way we may logically develop the subject and, at the same time, coordinate our experiences of nature. If we find that the conclusions of an ideal problem, correctly worked out, differ from the solution furnished by experiment, we must, if the "application" is to be maintained, alter the laws of our science. The ideal solution was correct but the "application" was wrong.

As matters now stand, we have therefore three subjects to consider: pure mathematics, physics (neither of which it is necessary to define here) and that ideal science for which I shall use the English term, applied mathematics. In the United States applied mathematics has been almost entirely treated as a department of physics. It has not been considered as a science in itself, but only in so far as it was useful to physics. The backwardness of its development here would seem to be largely due to this view. It has usually been taught by the experimental physicist, and the pure mathematicians have neglected it. And yet, as a science based on exact definitions and axioms, the pure mathematician is better equipped to teach its elements than the physicist, who is obliged to concentrate much of his attention on natural phenomena. However this may be, some reconstruction of our methods is necessary if we are to have our share in the developments which are taking place elsewhere. We cannot hope immediately to reach the ideal condition in which every worker in physics has received a thorough training in applied mathematics, but we can recognize the defects of the present system and make a start towards their elimination.

Professor Jeans has grappled with the difficulty in his book. It has hitherto been expected that in a single course on statics and dynamics the student should be led from the elements of the subject, at least as far as the applications of Lagrange's equations, and this without a very extended knowledge of the calculus. With the average student, whether his main study be physics or mathematics, this is scarcely possible. In most of our colleges and universities we recognize the necessity of first and second courses in calculus and one can hardly hope for equal efficiency in the study of mechanics without a similar method of procedure. If the school courses in physics were replaced by courses in elementary mechanics — the foundation of all physical theories — or if the same courses were laid out more with regard to the theoretical than to the experimental side, it would not be difficult to give the student a sufficient grasp of mechanics in a succeeding year's work. As a matter of fact, the experiments in a first course on mechanics can be almost entirely replaced by illustrations drawn from the ordinary phenomena of daily life in which every boy takes an interest, and he would learn much more surely the meaning and application of the laws of the subject than by using or looking at carefully constructed pieces of apparatus designed to illustrate special points. As long as the elements of mechanics are simply considered as an adjunct of physics and not as the basis of a science governed by exact laws, so long will there be an absence of any real development of applied mathematics. There are not short cuts to the higher branches of these subjects any more than there are to those of pure mathematics or of any other progressive science. The principles must not only be understood but thoroughly absorbed and used by the student for the solution of problems. As suggested above, this might be effected, partially at least, by inserting amongst the college entrance papers one on mechanics as the best introduction to the college courses in physics, the treatment being similar to that given in the numerous elementary text-books on statics and dynamics used in the English schools.

We have however, to deal with conditions as they are, and to consider how far Professor Jeans's book goes towards meeting them. It starts with the elements — kinematics and the laws of motion — and closes with the motion of a top, Euler's and Lagrange's equations, and Hamilton's principle. The main portion is undoubtedly a success in giving as rapid a development as can be expected. But in Chapters XI, XII the transition to the motion of a rigid body and to abstract dynamics is made with a jump which can hardly fail to leave the student behind. One receives the impression that these chapters have been inserted to satisfy the demands of the physicists, since their higher courses require the use of Lagrange's equations and generalized coordinates.

To come to details, the first chapter on kinematics is made a little hard for the beginner owing to the methods by which vectors and centroids are introduced; the mathematical parts of these might have been omitted or left until a later stage. In the second chapter, the laws of motion are clearly and *accurately* stated, and the deduction of the fundamental equation of dynamics

$$P = mf$$

(where P is the force which produces in a mass m an acceleration f) is made in such a manner as to show its importance for the whole subject. In the following chapter the experimental laws of extensibility of a string and of friction between two surfaces are stated, so that materials are at hand for the solution of many problems which bear a close relation to those found in nature. The selection of these problems is well made. The author has avoided the insertion of many examples in which the difficulties are chiefly mathematical, although he does not forget that the student has to be taught the application of mathematics to physical problems. One is only too familiar with devices for avoiding the use of mathematics — the laziness which takes the most pains.

Chapter V is somewhat difficult and might have been expanded, without adding to the length of the book, by cutting out some of the numerous determinations of centers of gravity given in the following chapter. The applications to problems of couples and forces in three-dimensional space are not easy to teach, while the idea of center of gravity is simple and is generally possessed by the student beforehand. One of the best expositions is that on work and energy, leading to a discussion of stability.

In the problems illustrating the motion of a particle under special forces, it is useful to find that motion under gravity is fully treated and that central forces, interesting as they are for the astronomer and the mathematician, are only given in sufficient detail to show how the laws of motion and that of gravitation are applied to the motions of the planets. Some advantage might have been gained by placing together the three principal methods of expressing the velocities and accelerations of a particle moving in one plane.

Fifteen years of attempts by the writer of this review to cover ground similar to that given in the first three hundred pages of this volume, with various text-books, occasionally without a text-book, and finally one year with Professor Jeans's book, convey a strong impression that the last, so far as he knows, is the best adapted to the needs of the American student under present conditions. That constant assistance and explanation will be required from the instructor is no defect; these are expected in the class-room, especially in a subject where the best results can only be obtained by outside work on the part of the student in the solution of problems and subsequent demonstration by the instructor. The few criticisms made here are on points not essential for the successful use of the book. Accuracy of statement is one of its chief and most pleasing characteristics. It is also well printed, and there is an absence of the too numerous varieties of type, designed like the headlines of a yellow journal to draw attention, which disfigure so many modern text-books.

The author is to be congratulated on a real advance towards the solution of the problem of introducing a first course in the teaching of mathematical physics in American colleges and universities.

ERNEST W. BROWN.

YALE UNIVERSITY.

ECONOMICS.

The Nature of Capital and Income. By IRVING FISHER. New York, The Macmillan Company, 1906. xxi + 427 pp.

The Rate of Interest. By IRVING FISHER. New York, The Macmillan Company, 1907. xxii + 422 pp.

ONE might almost say that, directly or indirectly, consciously or unconsciously, everybody is forced into intimate and vital relation with a large portion of the body of facts with which economics deals, and hence one might infer that the study of economics in one way or another cannot fail to be of interest to most rational beings. As an art, economics is of hoary antiquity; as a domain of thought, recognized, delimited, and dignified with a special name, it is relatively modern; as a science, it is but in its infancy. Strictly speaking, it comes into close relation with mathematics only as it becomes a system of deduction; and in so far as it may have advanced to the deductive stage, it is mathematics. Of the authors who have materially contributed to the inclusion of their chosen science of economics among the rapidly growing number of mathematical disciplines, Irving Fisher, himself at first a mathematician, is one of the most eminent; and his recent works in economics, The nature of capital and income and The rate of interest, although written in a style essentially adapted to the comprehension of a wide range of general readers, merit more than a passing notice at the hands of mathematicians.

To us who are professionally accustomed to deduction, who must intuitively feel a definition and a proof even when we do not explicitly state them, who instinctively seize upon an omission or an error in logic and immediately become perplexed and sceptical, the reading of the usual treatises on economics can give but slight logical satisfaction. What are the undefined symbols? What the postulates? What the inferences as distinguished from the presumptions? The authors do not say, perhaps they could not tell, very likely they do not appreciate what those words mean to us, or even feel aggrieved or scornful at our asking. In rare instances, as in the classic treatise of Mill or the brilliant chapters of Hadley, the style of the author will carry the reader along and force at least a temporary conviction, much as a fine poem will convince. It is really not a case of demonstration. An equally habile stylist taking a different, perhaps a contradictory view will produce the same effect. Then if the reader, at a loss between the two, should search for a foundation from which he might build up to sure conclusions to match against those of either side, he would find that the foundations were precisely the only missing parts of all these admirable structures.

So long as the literary attitude prevails, a good style will cover many an economic sin while uncovering a multitude of graces — a bad style will but uncover the sins and mask the graces. Fundamentally there is little difference. Fortunately economics is advancing toward the scientific state and some writers are forcing themselves to adopt a scientific style. We have only to turn the pages of such a treatise as Pareto's Cours d'économie politique or his article in our Encyklopädie to find a presentation of the subject with definite foundations and clear deductions therefrom. The premises may not be stated, perhaps as yet cannot well be stated, with the completeness and accuracy attainable in the theories of number or geometry — no more are the premises of mechanics or of physics — but the scientific training and attitude are there, and their accomplishments are already great. If I might be pardoned a personal observation, I would say that I had read or labored through much economic literature only to become the more bewildered until I came across the writings of Pareto. And now that it is seen that economics can be not merely clear and convincing (a matter of style) but logical as well (a matter of scientific method), perhaps there is still some hope for metaphysics or at any rate for epistemology. One after another the different domains of research seem to be creeping out of the twilight zone of campaign oratory.

One of the essential characteristics of the two books under review is this striving for definiteness of hypothesis and definition, for thoroughness of deduction. And that this fact has not been lost to the economists themselves may be illustrated by a few short quotations from as many reviewers. "The most scientific discussion of this subject in any language." "No treatment so lucid, so scientific." "It is characterized by a certain scientific hard-headedness which is not always found now-a-days in writings upon capital and allied subjects;" when was it? "It seems logically impregnable." "Lucid and tenaciously logical." "His concepts and definitions have all the exactness of a textbook on mechanics." This last, as we shall see, is essentially right; but unless its writer had in mind some one of a very few texts on mechanics, his statement will probably not appear to us so unqualifiedly a recommendation for logical perfection as he may have imagined. It is a pleasure to see that entirely apart from the recognition of the excellent economic doctrine which Fisher's new work contains, there is a general appreciation and approbation of the spirit in which the work was accomplished.

The first two chapters of The nature of capital and income take up the definitions of wealth and property-rights. According to the author, wealth and property (short for property-rights) are correlative terms; wealth being the concrete material thing owned, property the abstract right of ownership. These definitions are carefully explained and compared with the definitions or lack of definition of other writers. There is the frank acknowledgment that the definitions include more than is sometimes included owing to the fact that no specification that wealth or property shall be confined to exchangeable objects is made. As a partial justification, it is stated that: In definitions, it is usually better to include too much rather than too little. This generality has some glitter to it and might better have been omitted. What the author meant appeared fairly clear at first blush and in the surrounding context, but even the semblance of meaning vanished some twenty pages later when, to emphasize the word material in his definition, he stated: There is no advantage, but much disadvantage, in including any immaterial elements in wealth. This is perhaps a small point on which to raise an objection and it may be dismissed with the hope that readers will not notice it; logical purism is a very special art and science, and appeals little to any but the purist himself. The fact is that the author chooses to include in his definitions some things that are usually not included and to exclude from them some things which are not always excluded; and his choice seems to be happy. Wherever he confines himself to exposition and to marshalling facts, his style is a model of clearness and his array a marvel of tactics.

On pages 26 and 27 there is an illuminating table of a large number of articles of wealth with the correlative property-right. There is also given the name of the corresponding evidence or certificate of ownership. Thus a dwelling, the right of the tenant to use it, and the lease, or a railway, the right to a trip, and the ticket are examples of wealth, property, and certificates. A question, which the text does not seem explicitly to answer, arises to the mind. What is the certificate? In particular, is it wealth? Apparently it is material, apparently it is owned, apparently it *is* wealth according to the definition. Is then the evidence of the ownership of wealth to be classed as wealth itself? Let us take an example. Suppose I carelessly drop a gold double eagle from the deck of a vessel into the mid-Atlantic. A definite amount of wealth is apparently destroyed; for although the material remains, the ownership by any human being has disappeared. Suppose it had been a treasury gold certificate for $20 which had fluttered overboard. This material thing has been lost, my certificate of property is gone, but the wealth, namely, the treasury's reserve gold against the certificate, remains. Has any wealth been lost? If we answer No, then the certificate could not have been wealth except qualitatively: for the quantitative coefficient would be 0. If we answer Yes, then the loss is that of a rectangular piece of paper in which are imbedded a few silk threads and upon which are impressed sundry shades of ink. And this, I take it, contains the answer to the question. The certificate is wealth of a certain sort and represents wealth or a right to wealth of another sort.

After thus settling our question in a way which is perhaps satisfactory and perhaps not, a reopening of the whole matter appears advisable when the matter of price and value is taken up. To get at a definition of price the author restricts himself, at any rate at first, to wealth. If the quantity of wealth A is exchanged for the quantity of wealth B, the price of A in terms of B is the ratio B/A and the price of B in terms of A is the ratio A/B. Price is therefore a ratio of exchange. Value is the product of price by amount. Price and value as applied to property-rights do not seem to be defined so clearly, but probably all that is necessary is to substitute the word property in the place of wealth in the previous definition. In fact it is doubtless necessary to be able also to substitute partially and thus to have the ratios wealth/property, property/wealth, and property/property in addition to wealth/wealth. It is now interesting to return to the consideration of certificates. The details of the discussion will be omitted. Attention may, however, be called to one of the conclusions, namely, that, technically speaking, value is not a distributive operator even for infinitesimals. For the certificates, being wealth and being constantly exchanged, must possess price and value, whereas the value of a certificate and of the wealth to which it certifies the property-right is by no means to be counted as equal to the value of the certificate plus the value of the wealth represented; symbolically, $v(c+w) \neq v(c) + v(w)$.

The third chapter treats of utility or ophelimity or desirability or whatever one wishes to call that important economic concept. It is pointed out that the desirability corresponds to the fact that some one desires something, without considering the question as to whether he should desire it. The term is to be applied to such "noxious things as opium, alcohol, and degrading literature" — rather bad company for an article of commerce which is among the most useful in the arts, sciences, and industries. Thus far the definitions have been merely preliminary. Capital and income are introduced in the next chapter. The new definitions are: A *stock of wealth* existing at an *instant* of time is called *capital; a flow of services* through a *period* of time is called *income*. Subsequently this capital is designated as capital-wealth or capital-instruments and further definitions of capital-property and of capital-value, the latter being the value of the capital-wealth or capital-property, are given in chapter five. As it does not appear that the words *stock of* prefixed to wealth add anything to the conception of wealth, it seems that capital-wealth is identical with wealth. The important idea is that the wealth is as it exists at a definite instant of time. In mechanics nobody would think of defining mass as material objects and then redefining by saying that capital-mass was a stock of mass existing at an instant of time; but in economics the word capital has stood for so many things or has been used so much without standing for anything definite that the author has to exercise great care, considerable repetition, and a good deal of rephrasing and redefinition to accomplish his ends. He seems to be taking more care than is frequently expended on definitions in mechanics.

The chapters on capital accounts and capital summation are especially valuable in bringing out the connection of the theory of capital with business usages. And the advantages arising from the comparison are not by any means one-sided; added clearness and practicability for the theory go hand in hand with useful illumination as to what are or what might be good methods in accounting. The insistence on relating theory to practice in the treatment of capital and later in the treatment of income furnishes several chapters which are largely novel with the present author. In the course of the investigation there is outlined a double method of accounting, one the method of balances, the other the method of couples; the first gives the distribution of capital among different individuals, the second catalogues the things owned. Applications are given to various points such as the accounts of a real person, the accounts of a fictitious person or company, the matter of taxation.

An extended chapter on income is required to make clear the author's definition and to justify it. To define income as the services of wealth forces the inclusion into that concept of a number of items which are frequently omitted, and the inclusion seems convenient. The development of the subjects of income accounts and of income summation follows. Again the double method is employed. Again there are numerous applications to questions of interest. Frequently the applications are merely hinted at and left either for the reader to think out or for the author to recur to at a later point. There is also a chapter on psychic income; for to stop short with material income is not wholly satisfactory. By carrying the method of income summation by couples to its logical conclusion, it is seen that no objective items of outgo survive cancellation and that in a comprehensive view of production there is no such thing as cost of production in the objective sense. As we thus ultimately arrive at the necessity or convenience of introducing the psychic element, the query naturally arises as to whether it might not have been more strictly logical to begin with the psychic and introduce the theory of desirability more fundamentally into the work rather than to give so much preliminary space to the material conception of wealth. To have done so, however, would have rendered the book much less readable and might have seriously interfered with its function as a mediator between theory and practice.

The next great section of the book contains what is really the author's main contribution to the theory of capital and income. Nowhere, however, are the essential elements of his theory so vividly and graphically set forth as in the tabular diagram which is taken from the subsequent book on the rate of interest and which may appropriately be explained at this point. The idea is this. There exists at any time a stock of material objects in the world which are owned by human beings. These objects will or at any rate are expected to yield services to human beings during a more or less extended period of futurity reckoned from the instant at which the stock is evaluated or rather surveyed. These services may be material in the first instance but ultimately they are psychic. The reason we want the wealth, the reason we put ourselves to pains to acquire the wealth or some property-right in it, is that we are desirous of obtaining the satisfactions due to these services. The income concept, the natural prevision of the human being for the future whether immediate or remote is the vital psychic element in the present theory. Now it is the income, the services from the wealth, which possess value. And finally it is this value which determines the value of the wealth or capital

	Present capital	Future income
Quantities	Capital-wealth →	Income-services ↓
Values	Capital-value	← Income-value

itself. Without expected services capital would be fruitless, it would be valueless. The actual value which is to be attributed to capital depends not alone on the value of the expected services or income but on the rate of interest.

With this perspective of the general theory in mind, we may return to the systematic outline of the text. The ratio of the value of services obtained per unit of time from a certain capital to the value of the capital is designated as the value-return. The rate of interest is then defined as the value-return provided the income is perpetual and flows at a uniform rate. In this procedure it is clear that the author is not yet following out his theory but is merely stating a point of view which is to be abandoned. After some explanatory considerations which are set into intimate relation to investment usage, the foregoing definition of the rate of interest is rephrased as: In the *price* sense the rate of four per cent per annum means that the price of $100 of capital is $4 of income per annum for ever. Next the premium sense is introduced with the statement: The rate of interest of four per cent per annum in the *premium* sense means that the price of $100 of goods at a given date is $104 of goods at the same date a year later. It is then shown that if the rate of interest in either of these senses is constant, it is so in the other and the two rates are equal. If the rate is not constant the two senses give different rates, and the author finds it more convenient to adopt the premium sense as fundamental.

After some words on discount and many an illustration drawn from the world of finance, the author comes to the conclusion or redefinition that the value of capital at any time is the present value of the total expected income discounted according to the rate of interest or rate of discount. One thing that impresses itself on the reader is the obvious difficulty that would have been encountered in developing the present theory in an age in which our financial system was only in a rudimentary form. Conceivably the theory might have been developed any time by any person who wished to take the income concept as fundamental and evaluate present goods in terms of the expected services therefrom, but it would have taken a prodigious genius to do it in the days when it was not safe to look more than a few hours into the future and when one would discount a kingdom for a horse. Even at present it shows great originality, when one can break away from the classical conception of contemporaneous barter and found one's economics on the future and on a universal system of credit. The next, the fourteenth, chapter is one of the most interesting in the whole work; it is the one which above all others clinches the theory. In it is discussed the matter of earnings and income. Evidently earnings may exceed or fall below income: for one may encroach on his capital or he may allow it to accumulate. Now, it is very tempting to consider earnings and not income as determining the value of capital and to rank savings as capital. The theory is intricate and consequently the exposition is detailed and replete with illustrations. In the end the reader will be sure to find himself convinced of the author's honesty and correctness, and he will probably recall the bluff remark of the hard-headed old business man who exclaimed "Who ever heard of dividends doing a stock any good!" The financial injustice of taxing earnings instead of income — and the moral injustice, too — in that the conservative and saving person is unduly burdened relatively to the spendthrift, is sharply exhibited.

To complete the work and offer, so to speak, a second approximation to reality there is given a chapter on the element of risk. The simple presentation of the theory of chances, the introduction of the coefficients of risk and of caution, and the application of these matters to determining the value of capital, for instance stocks and bonds, will be of interest to investors and brokers. No treatment of the risk element could be considered as complete without some more or less extended mention of speculation and the use of speculation to the public at large. Most of the author's statements on this subject seem to be in accordance with the best accepted views; once the risk element has been introduced, his own theories seem to have relatively little bearing on speculation as such. There are two points at which it seems as though the text implies doubtful if not erroneous views. In speaking of the fact that much of the evil of speculation arises from the lack of independence of the views of those who participate in speculation and with especial reference to the participation of the public in speculative markets the author says: How easily they are led is shown by the effect on the stock market in the year 1904, when Thomas Lawson published scare-head advertisements in the newspapers advising the public to sell certain securities. This seems to imply that it was the selling induced by Lawson's advertisements which caused the sharp reaction, sometimes (especially by Mr. Lawson) called the Lawson panic, of December 5–12 in 1904. Maybe it did; but we doubt it and believe that the author would have been on much safer ground if he had pointed out that stocks had advanced steadily from the middle of May till the first week of December carrying the average of twenty rails up 26 from 93.5 or toward thirty per cent and of twelve industrials up 26 from 47.5 or toward sixty per cent,* that in all this time there had not been a reaction of three points in the averages, that after the middle of October the daily sales including Saturdays had averaged well over a million and a quarter shares in the New York market, that such a condition almost certainly represented undue speculative activity and growing overconfidence and a large weakly margined bull account, that in the preceding month there had been a pronounced bulge — in short that everything was shaping up for a violent and drastic shake-out. It is by no means certain Lawson did not time his advertisement with the drop rather than that the drop resulted from the advertisement. It is even conceivable that the advertisements lessened instead of accentuated the inevitable break; stranger things have happened. Perhaps the author knows what went on behind the scenes and may therefore be right; but his statement unfortunately sounds like the meaningless "causes" ground out in haste and from hearsay by the underpaid so-called financial reporters.† The other remark to which we would take exception is: Normally speculative property will gravitate into the hands of those most able to forecast its true income. This is another generality with something of a glitter. It is difficult for us to see why anybody who is best able to forecast the true income of a property should care to allow that property to gravitate into his hands if his forecast leads him to believe that the true income is shortly to be nil. If Homer has nodded, it may be forgiven.

Following the conclusion of the main part of the work there is an elaborate summary of it with ample diagrammatic representation and also a further general summary of the whole book. There is an extensive glossary of definitions with a reference to the appropriate passage of the text. Upon this is added a series of appendices to the various chapters. These appendices contain a variety of matters most of which are of a mathematical nature. The mathematics, however, is not so much that of economics in Pareto's sense as of accounting and interest reckoning under various assumptions, and as such it will probably not appeal very strongly to the mathematician. An exhaustive index concludes the treatise. From the detailed style of the text and the frequent summaries the reader can hardly fail to see clearly what the author's theory is and by the glossary and index he can readily turn to any part of it which he may desire to reread or study with particular care after completing or during the progress of his perusal of the work. Thus to aid the student is always a kindness on the part of any author; it is almost a necessity when so much that is new in theory and novel in illustration has been presented.

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* *Wall Street Journal* figures.

† It may be noted that the explanation here offered as to this panic is just as valid for illustrating and enforcing the author's point as his own; it is merely more subtle and less obvious — in Wall Street the obvious is seldom right.

The general arrangement of The rate of interest is similar to that of The nature of capital and income; there are the extended analytical table of contents, the careful summaries, the glossary of definitions, the appendices with mathematical complements of the same sort as before and with a vast array of statistical information, and finally the comprehensive index. In the preface by way of explanation as to why the mathematics is relegated to appendices the author states that: Mathematics can properly claim no place in economic discussions except as they add something not expressible, or at any rate only imperfectly expressible, in ordinary language. Perhaps this is equally true of discussions in mechanics and physics and chemistry, but mathematicians of the present generation might wish to put in a counter claim by rejoining that verbal expositions in any subject are unnecessary except in so far as they serve to explain the meaning of the mathematical results obtained or to treat portions of the subjects which must still be regarded as matters of opinion with the author instead of matters of deduction.

As the present author has an original theory to propound, he devotes nearly a quarter of the main text to a criticism of previous theories. First come the crude theories, then the productivity theories followed by the cost theories, and finally Böhm-Bawerk's theory. To the last, as the most elaborate and thoroughgoing of recent theories, the author gives the larger part of his attention and in this he has had the advantage of the criticisms of Böhm-Bawerk himself. Throughout this critical introduction there is no attempt to throw out all the elements of all preceding work on the subject of the rate of interest but rather is the effort to eliminate that which is mischievous while retaining that which is good. There is, indeed, here and throughout the book, much reference to and many quotations from John Rae, to whose memory the work is dedicated and on whose theories the author has built his own. We shall take no time in discussing this which is itself a discussion of the work of others, but shall pass on to the remainder of the text.

The great complexity of our social and financial organism requires the investigations to advance by successive approximations and the author divides his presentation into a first, second, and a third approximation. Preliminary to the first there is a chapter on appreciation and interest. It was seen in the course of the remarks on the last book that unless the rate of interest is constant it is not the same when expressed in different senses as a price and as a premium. In like manner here, if the price of wheat is rising year by year relatively to the price of gold, one could afford to exchange $100 now for $104 to be paid a year hence as easily perhaps as he could afford to exchange 100 bushels of wheat now for 102 bushels to be paid in a year. If commodity prices in general are rising with great rapidity the rate of interest in money may have to be very high in order to keep the rate of interest in commodities above zero. We see, then, what a delightful disregard for economic law our legislators have when they limit the interest rate in money to six per cent. If commodity prices advanced about five per cent from 1905 to 1906 and about as much from 1906 to 1907, then one who loaned at the maximum legal rate during these years was realizing only about one per cent and those who had their money in a savings bank were losing about one per cent. As a matter of fact the advance was, if anything, more rapid. There is another and highly important preliminary chapter on the time-preference, or the willingness of any individual to do without income now for the purpose of having (more) income in the future. The dependence of the time-preference on the expected distribution of income in time is also discussed. Thus if a student is obtaining his education and therefore has reason to expect that his income will be much enhanced in the future he may be perfectly justified in borrowing money even at a considerable rate of interest, for he needs it now and can spare it later — in this way he may solve the problem of getting the most out of his income in the long run; but if one looks forward to a diminution of income he may be willing to loan now even at a low rate of interest for the purpose of having income when he needs it more — and in this way he may solve for himself his own maximum problem. Throughout this chapter the text is especially full of delightful and valuable illustration.

We now come to the seventh chapter, the first approximation to the theory of interest, in which the assumption is made that the income is rigid. The chief concern of this chapter is to show how an income which is originally definite and rigid may be modified by the simple process of loaning or borrowing according to the individual's time-preference and the current rate of interest. The amount borrowed is regulated by the theory of marginal desirability. For if one's time-preference is ten per cent and the market rate is five, he will start to borrow and as he borrows his time-preference diminishes until after he has contracted loans for a certain sum his time-preference has sunk to the market rate; and incidentally unless the available supply of money is infinite, the market rate will be raised. The matter is similar in the case of making loans. Hence if the supply be now considered finite in relation to all the borrowers and lenders, it is possible to get at a determination of the rate of interest by the following principles: First, the rate of time-preference of each individual for present income as compared with remoter income depends on the character of his income stream as modified by his borrowing and lending. Second, there results a state of equilibrium between the time-preference of all individuals wherein those preferences are all equal and equal to the market rate. And third, the market rate is thereby determined as that rate which will throughout each period of time clear the market. We have here what appears to be a perfectly good definition of the rate of interest wholly determined by the statement that an economic state of equilibrium is established.

It will be well to stop a moment and review the logic of the theory. Let us assume with Pareto the existence of the homo œconomicus or the class of homines œconomici. That which these ideal individuals desire are satisfactions spread throughout a period of time, that is, they want income. Some may have more income than they want this year and some may have less than they want. They therefore set about bartering current income for future income until each is satisfied with his current condition as compared with his future prospects and a state of economic equilibrium is reached in which each homo œconomicus will pay or accept the same premium as any other of his species for the exchange of present and future income. The premium has a definite ratio to its corresponding income, namely, the market rate of interest. Then any income *I* may be capitalized into the capital *C* by the equation $C = I/r$, where r is the rate of interest. In this way it would appear as though wealth and property were eliminated from the really vital part of the theory. Indeed it seems as if there were no essential need of the lengthy discussions as to the materiality versus the possible immateriality of wealth and as to the exact correspondence between wealth and property-right. And in view of the fact that we ultimately come to consider psychic income as the only fundamental income, those discussions impress us as rendering the theory more rather than less obscure from the purely logical point of view. But even if it be granted that we are not amiss in these statements, few would seriously contend that the author has not adopted that presentation which will most appeal to the largest number of persons not technically interested either in logic or in mathematics.

Now that we have acquired an insight into the essential elements of the author's theory we must unhappily run somewhat cursorily over the large amount of material which remains or we should prolong this already lengthy review far beyond the limits which may properly be assigned to it. The second approximation regards income as flexible and has important applications of a practical nature to the question of how one may vary his income by choosing among the many different optional forms of investment. An investment in forest lands will clearly give a different distribution of income in time from that obtained by investing in farm lands or mining property. The author gives a special chapter to the consideration of the various classes of options. There are also unknown options which may arise during the course of time by the introduction of new inventions. The discussion of these naturally leads over to the general question which constitutes the third approximation under which the income is treated as uncertain. On page 221 there is an interesting table in which the conditions under the different approximations are compared.

There remains about one hundred pages of the text, which bear the general title Conclusions. Herein are found, among other things, some detailed applications of the completed theory to actual conditions, an elaborate inductive verification of the theory from the monetary point of view, an equally elaborate inductive verification from the economic point of view, and the inductive refutation of the so-called money theory of the rate of interest. One of the most interesting applications or verifications of the theory which Fisher sets forth is the discussion of appreciation and interest. If commodity prices are increasing, it may be assumed that money is depreciating and vice versa. As we have seen, a depreciating monetary system ought to imply a high rate of interest part of which may be applied to the amortization of the lessening purchasing power. The text discusses this matter in detail and presents the conclusion that theory and past experience go hand in hand. We have constructed a chart — it is too bad the author did not construct one — which exhibits the average interest rate of the Bank of England from 1846 to 1907,* the annual index number for English commodity prices over the same period, and the annual percentage rate of increase of the world's visible supply of gold from 1851 to date.† Some words on the comparison of these graphs may be worth while.

It appears that from 1850 to 1857 commodity prices rose rapidly by about thirty per cent and interest rates rose sharply from under two and a half per cent to over six per cent. In the meantime the average annual increase in the world's visible gold supply was about three and a half per cent. There were unmistakable evidences of gold inflation, great apparent prosperity, and finally an international economic crisis. What the course of commodity prices and interest rates might have been in the years 1858 to 1872 if there had not been such a destruction of wealth by war, it is difficult to say.‡ As it was, prices remained nearly on a level and interest rates fluctuated widely while maintaining a high average over the whole period. Meanwhile the average annual increase of gold had fallen to about two and a half per cent. From 1873 to 1896 prices fell away from the high mark 111 to the low of 61. If this is to be attributed to appreciation of gold, the appreciation was really

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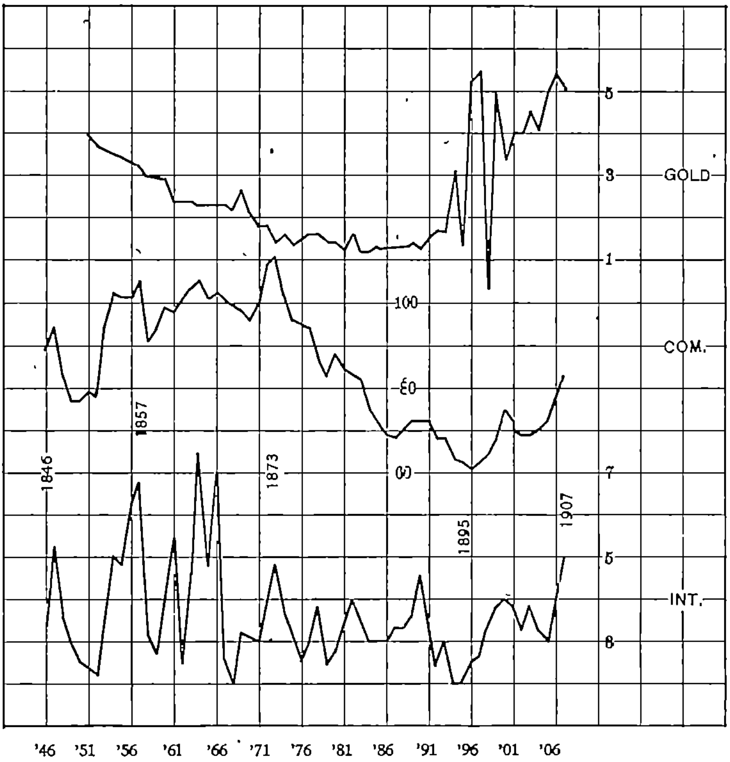
* The table given by the author on page 418 apparently contains some misprints in the earlier years.

† It is very difficult to get a reliable estimate of the visible supply from year to year, and the annual addition to the supply may be almost anything between zero and the total amount mined, depending largely upon the competitive demands of the money market and the arts and industries. The chart must therefore be read without the use of a micrometer.

‡ There seems to be a generally accepted theory that war destroys wealth and is therefore equivalent to a call for more gold. This appears to be a curious theory in view of the fact that gold is certainly not destroyed by war, earthquake, and the like. Would it not be more reasonable and more in ac-



Upper curve. — Percentage annual increase in the visible supply of gold; scale on the right.
Middle curve. — English commodity prices; scale in the center.
Lower curve. — Average interest rate of the Bank of England; scale on the right.

remarkable. At any rate interest was low and for two years at the end of the period was at two per cent. It will be ob-

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cord with the theories of these books to claim that such calamities were equivalent to an increase rather than a decrease in the gold supply? The whole matter, however, is so complicated and so bound up with the question of credit that it cannot be answered by any off-hand query or reply.

served that the fall in interest was not continual as in commodity prices. The exigencies of the money market may change within fairly wide limits within tolerably short periods of time, and it is these rapid changes which must be smoothed out before the relation of appreciation to interest can be even approximately determined. In fact while prices were falling most rapidly, the rate of interest was swinging about the value three and a half. During all this period the percentage increase in the gold supply was also low and equal to about one and a half per cent.

If, then, one may assume that the long-time variations in prices and rates of interest are determined largely by the question of appreciation, it seems clear that an annual one and one half per cent increase in the gold supply was by no means sufficient to keep pace with the increase of the world's population, the rapid development of new country, as in Africa and North and South America, and the increase of business; it might have fallen still further short if the world had then been so unanimously committed to the gold standard as at present. Since 1895 commodity prices have risen sharply and interest rates have advanced. Coincident with this is a violent jump in the annual percentage increase in the world's visible gold supply; the present average of that rate may be taken as a little under five per cent. The author's theory seems to fit pretty well with the facts of recent history; and it is only history which is recent as economic cycles go that is available for a detailed statistical comparison with the theory. At present there are large numbers of financial writers and of economists who have a leaning toward the practical side of their subject who are crying gold inflation with all their might, are predicting very sharp advances in commodity prices, much higher rates of interest, lower prices for bonds and other securities with a fixed rate of income, great speculative activity, and finally, in perhaps five or six years, a panic as in 1857 and 1907. This would appear to match very well with Fisher's theory. The next decade promises to be an interesting affair. The world's visible gold supply is now estimated at something like $7,200,000,000 and we are adding to it at the rate of about 5 per cent per annum (the actual amount mined is about six per cent), which seems high. Unless we mine a progressively increasing amount each year, the percentage increase per annum in the supply will surely diminish; it will diminish unless the mining rate increases more rapidly than it has in the last three years. It is by no means impossible that a somewhat smaller annual percentage increase in the visible gold supply or even the present high rate may not be required for the proper opening up of new territory, for the increasing business and population of the earth, for the modernization of the civilization of such a country as China, and especially for the imperative fortification of the gold reserves of our own and other countries.* It may well happen that 1907 and not some later year will ultimately be compared with 1857 as containing a great panic due to gold inflation. If the fact that gold is depreciating is widely recognized and allowed to influence speculation and prices according to sane economic laws, the disastrous effects which would follow ignorance will be much mitigated.

In closing this review we would express the hope that not only economists and mathematicians and business men may read these volumes — it is almost essential to read both, so closely are they interrelated — but that those of our actual or would-be legislators who may be interested in something more than holding their ears to the ground to catch a rumble that will indicate some wild popularistic "vital" issue upon which they can conduct a campaign at least orally successful, will take the pains to study Fisher's work in detail, that they may legislate with as much wisdom as possible; we are in great need of a decent currency system.

EDWIN BIDWELL WILSON

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
BOSTON, MASS., *October*, 1908.

SHORTER NOTICES.

Vorlesungen aus der analytischen Geometrie der geraden Linie, des Punktes und des Kreises in der Ebene. By OTTO HESSE. Vierte Auflage, revidiert und ergänzt von S. GUNDELFINGER. Leipzig, Teubner, 1906. 8vo. viii + 251 pp.

THIS little book of the great geometer is so well known that it would seem almost absurd to review it at any length. For a generation it has been considered a model of elegance, and the new edition is, except for very minor changes, a reprint of the old. The editor has however added a number of notes, covering twenty-two pages, in which certain of the more difficult points find a fuller explanation. Special attention is paid in the notes to the matter of signs. One of them amplifies in a very desirable way the admirable treatment of involution given by the author. Most of the others will also be found to add materially to the usefulness of the book.

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* On the question of the gold supply reference may be made to Thomas Gibson's special market letters on "The Increasing Gold Supply," New York, 1908. Of the contributors to this symposium, Muhleman alone seems to regard the present supply as possible of absorption without serious inflational effects.

One finds scattered throughout these lectures little side remarks which merit special attention. For instance in the second lecture the author shows how by means of the method of abridged notation certain theorems about lines determined by triangles may be proved without any calculation. He compares this method with the elementary but laborious methods which make no use of special devices, but proceed upon the basis of the general principles alone. He then says: "Man braucht aber nur einmal solche Rechnungen mit den allgemeinen Prinzipien durchzuführen, um die Lust an weiteren Versuchen zu verlieren." He continues as follows: "The general principles, to be sure, indicate the path which is certain to lead to the desired result, but not the simpler way which one ought to choose in a given case. The amount of mechanical labor required by the general principles becomes greater as the theory grows more extensive and comes to include a larger circle of problems. It becomes necessary therefore to build up special theories within the general theory, each of these special theories being especially adapted for a certain class of problems. It is to these special theories that these lectures are devoted."

One other remark made by the author is characteristic of the book. He finds himself discussing, for the third time, the problem of orthogonal substitutions, using on this occasion a purely algebraic method. He says: "There can be no doubt that the purely algebraic problem gives rise to difficulties which did not present themselves in the previous methods of solution. This makes it so much the more important to accomplish the algebraic solution of the problem which from another point of view is already solved. It is precisely problems of this sort which lead to the most fruitful investigations." The perfectly symmetrical character of the book, in which the geometrical and algebraic problems are treated with equal care and so as to completely elucidate each other, is the main charm of Hesse's work.

I cannot close these remarks without expressing a bit of scepticism upon one point. The preface begins with these words; "The present treatise is intended as an aid to the study of geometry in school as well as at the university." It does not seem possible to me, even with the most generous concessions as to the superior abilities of the German schoolboy, to think of Hesse's book as a text for a beginner in analytic geometry. To be sure it requires no prerequisites other than those which the beginner in this subject is likely to have, but it does require a mathematician of some maturity to read it appreciatively. Certainly the contrast between this book and our present-day textbooks for beginners is great.

E. J. WILCZYNSKI.

A Bibliography of the Works of Sir Isaac Newton Together with a List of Books illustrating his Works, with Notes, by GEORGE J. GRAY. Cambridge, Bowes and Bowes, 1907. 8vo. viii + 80 pp. 1 plate. Second edition, revised and enlarged.

IT is now some twenty years since Mr. Gray issued the first edition of this work, but as only a hundred and twenty copies were then printed it was never generally known to scholars, save by name. In a way this has not been without its good results, since the very fact of its rarity has led to the preparation of this new and enlarged edition, containing information now published for the first time.

The work contains four hundred and twelve titles arranged under ten heads: Collected editions of works, the Principia, Optics and optical lectures, Fluxions, Arithmetica universalis, Minor works, Chronological, theological and miscellaneous works, Reports on coinage, Works edited by Newton, Memoirs. The bibliography is preceded by an analytic table of contents, and followed by an index of names. There is also a reproduction of Mackensie's drawing of Roubiliac's statue in Trinity College, from Le Keux's engraving.

Only one who has worked in the bibliography of mathematics can appreciate fully the extent of Mr. Gray's labors and sympathize with his errors and omissions. Such a work is easily conceived and can be pursued with little difficulty to a vaguely defined point, after which titles and editions become exceedingly elusive. It is like collecting portraits of Newton; the first fifty can be found with little trouble; the second fifty are to be had if one will give the time and money; the third fifty come very slowly, and the collector is increasingly conscious that some will always elude his grasp.

It is therefore no adverse criticism of Mr. Gray's labors to call attention to a few omissions, and to say that any bibliographer can easily suggest others. It is rather with the desire to help future editors that the following supplementary notes are added: Of the Mathematical Principles (No. 26) there was also an edition in 1846; of Frost's Principia (No. 37) the edition of 1880 might be mentioned definitely; Addison's Defense, in Fontenelle's Plurality of Worlds (No. 41), also appeared in 1801; Martin's Philosophia Britannica (No. 115) also appeared at London in editions from 1759 to 1787; there was a second edition of Snell's Newton und die mechanische Naturwissenschaft (No. 151) in 1858; there was also an edition of Voltaire's Elémens de la philosophie de Neuton in his Oeuvres, Neuchâtel, 1772; there was an edition of the Lectiones opticae (cf. No. 180) published in London in 1729; mention might be made of Alagrotti's Dialoghi sopra l'ottica Neutoniana, in his Opere of 1778–84, volume II, pages 1–292 (cf. Nos. 194–197); mention might also be made, after No. 218, of A. Johnson's article on Newton's use of the slit and lens in the *Transactions of the Royal Society of Canada*, 1892, volume IX (3), page 49; Lecchi's edition of the Arithmetica universalis (No. 280) appeared in 1752, and since the dedication is also dated 1752 this, instead of 1732, is probably the date of the first edition; after No. 372 it would be well to insert a cross reference to the Vita by Castillioneus mentioned in No. 2, and similarly after No. 378 to Chittenden's Life in the 1848 edition of the Principia (cf. No. 26), and after No. 379 to Davis's Life (27 pages) in the London edition of 1819 of the Mathematical principles; there was also an edition of Frisi's Elogio (No. 393) in 1829–30; the omitted date of Paulian's Treaty of Peace between Descartes and Newton is 1763; to the biography might be added a ten-page folio without date, Vita d'Isacco Newton scritta da Giovanni Batista Giramonti, but there are probably dozens of brochures of a similar nature. It would also seem proper to add Charles Taylor's Geometry of Kepler and Newton, published in the *Transactions of the Cambridge Philosophical Society* in 1900, volume XVIII, pages 197–219. Of course G. Kowalewski's Abhandlungen über die Quadratur der Kurven, Leipzig, 1908, appeared too late for insertion, although by its reference to Newton it might deserve a place. In the index the reference after Durkik should be 252 instead of 352. In addition to the errors in the first edition of the Principia, mentioned in Nos. 6 and 7, it may also be noted that the woodcut on page 22 is upside down, in some copies at least, as in one owned by Professor Hallock. The book is clearly printed and well indexed. It is needless to add that it deserves a place in every mathematical library.

DAVID EUGENE SMITH.

A Scrap-Book of Elementary Mathematics. Notes, Recreations, Essays. By WILLIAM F. WHITE. Chicago, The Open Court Publishing Company, 1908. 8vo. 1 plate. 248 pp.

IT is the right of every author to ask that reviewers judge his book only by the standard that it professes. If it pretends to be an exhaustive treatise, then the critic may justly claim that it falls short of what its readers have a right to expect if it contains more than the allowable maximum of errors or if the author displays ignorance of the work of his chief predecessors. If, on the other hand, it pretends to little, then it is proper to lower this standard, reserving the right to criticise the writer for not using his talents to better purpose. These or similar thoughts will probably occur to more than one reviewer who lays down Dr. White's readable little book after spending the short time necessary to enjoy its contents. Pretending, as its name implies, to be merely a scrap-book, and written only for high-school pupils or for tyros in the teaching profession, it is by its very nature immune to serious criticism. Given a few well-known histories of mathematics and books of recreations in the same domain, such a work is easily constructed. If it has errors, these are attributable to the source material; if the bibliography is meagre, the nature of the book does not warrant one that is more extensive; if the selection is not the absolute best, the readers to whom the book appeals are all the more pleased; if the arrangement shows little system, this is only what may be expected of a "Scrap-book of mathematics."

Some idea of the scope of the work may be obtained from the following list of certain of the topics treated: Numeration of large numbers, Numerical curiosities, Tests of divisibility, Miscellaneous notes on number, Numbers arising from measurement, Compound interest, Arithmetic in the renaissance, Geometric puzzles, Magic squares, Alice in the wonderland of mathematics. These and others have been treated briefly and in a style that will entertain and somewhat instruct those for whom the book was written, so that the work will serve its purpose as a not unworthy addition to a high school library. That it contains little that is not found in such books as Ball's Mathematical Recreations, Lucas's Théorie des nombres, and a few other less valuable works, is not a serious objection to such a publication.

As to the errors in the book, these are chiefly, and by the nature of the case, historical. They arise from the fact that Dr. White has not always depended upon authorities of recognized standing to-day. For example, the date of Paciuolo's *Suma* is given as "1494 (some say ten years earlier.)" Now we know perfectly well that the first edition of this famous work appeared in 1494, at Venice, and we know when the book was written, so that all element of uncertainty has long since been removed. Under the use of the period as a decimal separatrix, mention may now be made of the Pellos arithmetic of 1492, although when Dr. White's manuscript was prepared this use was unknown. That Avicenna, or Ibn Sînâ, was the first writer on arithmetic to use the Hindu numerals is impossible, for he wrote nearly two centuries after Al Khowarizmi had made them known in Bagdad. It would hardly be expected that Cardan, writing about 1550, could be taken as a present authority on the introduction of the Hindu numerals in Europe; indeed we now know that Fibonacci was by no means the earliest one to know them in Italy, as he asserted. As to the lack of a suitable notation for computation among the Greeks, it should also be said that Dr. White's assertion is now felt to be too sweeping, and that the Greeks really got along very well with little practical use of the abacus. That 'million' first appeared in print in Paciuolo's work (1494) is not correct; it is found in the Treviso arithmetic of 1478, in Borghi's work of 1484, and in the Pellos book of 1492. The word was already old when printing was invented, and appears in English in Pier's Plowman as early as c. 1375, coming from such Italian writers as Marco Polo. That Parley, in 1596, was the author of the classic "Thirty days hath September" has long since been disproved. It appeared in English at least six years earlier, and was printed in Latin as early as 1488. It is mediaeval. The statement about the sine is not exact. Hultsch has shown that the Greeks also used the half chord, and

Gherardo of Cremona probably used the word sinus before Plato of Tivoli. As to the symbol for infinity ascribed to Wallis, a rather hasty examination of his Opera fails to show that he used the one mentioned, but he mayhave done so.

It is perhaps justifiable to revert to one point of criticism already mentioned, and to inquire whether Dr. White has used his talents to their best advantage in planning and executing such a work. There is an undoubted need for a good book on mathematical recreations arranged more systematically than any that has yet appeared; one that shall place the material in somewhat the order that a teacher or pupil may use it, and that shall rather carefully exhaust the available material in the large number of works on the subject. To write with no extensive knowledge of the best modern works on the subject, such as Ahrens's and Schubert's, and of such noteworthy older works as those of Bachet and Ozanam is to render impossible the accomplishment of such a labor.

It must be repeated, however, that the author makes no pretense to this sort of work, and it is only fair to say that he has given us a very readable book for a summer afternoon.

DAVID EUGENE SMITH.

Royal Society of London Catalogue of Scientific Papers 1800–1900. *Subject Index, Volume* I, *Pure Mathematics*. Cambridge, at the University Press, 1908. lviii + 666 pp. Price $6.75.

THIS is the first volume of a subject index which is to be published as "separate index-volumes for each of the seventeen sciences of the Schedules of the International Catalogue, viz., mathematics, mechanics, physics, chemistry, astronomy, meteorology, mineralogy, geology, geography, palaeontology, biology, botany, zoology, anatomy, anthropology, physiology, and bacteriology." The object is to bring this Index into close relation with the International Catalogue of Scientific Literature by adopting the same general method of classification and by indexing the papers of the whole of the nineteenth century, while the International Catalogue is devoted to the literature following this period. The present Index has also close contact with the well known Catalogue of Authors which is being issued by the Royal Society, and of which twelve large volumes (1800–1883) have been published, while those covering the period from 1884 to 1900 are in preparation.

As may be inferred from the title, the scope of the Index under review is not as extensive as that of the International Catalogue, since the former confines itself to periodic literature while the latter includes also the non-periodic publications. Notwithstanding this limitation, the preparation of such an Index for what has been called the historic century * is an enormous undertaking in view of the fact that the headings of articles are frequently too vague to admit of classification without a detailed examination of the contents of the articles. We are told in the preface that the present volume contains 38748 entries referring to 700 serials. Although these numbers may appear large, yet they are too small to imply a practically complete list of the papers which appeared during the period under consideration. In fact, about 950 journals devoting more or less space to mathematics were started during the nineteenth century and over 200 had their origin in the preceding century.†

One of the most important mathematical periodicals which does not appear in the list of serial publications used in preparing the present volume is the *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, founded by Hoffman in 1870. Among the American periodicals which have not been indexed, the *American Mathematical Monthly* is probably the best known, and I found only two of the early American mathematical periodicals mentioned by Hart, volume 2 of the Des Moines *Analyst*, in the given list of serial publications, viz., *The Mathematical Miscellany* (1836–39) and Runkle's *Mathematical Monthly* (1858–61). Some of these early periodicals were devoted entirely to problems and hence did not deserve a place in the list under consideration, but this is not true of all of them.

Although the number of entries and the number of serial publications used are not sufficiently large to justify the hope that the volume under review might contain a practically complete index of the mathematical papers appearing during the nineteenth century, they are large enough to inspire considerable confidence in the value of the Index. When it is remembered that Herr Valentin estimates the total number of mathematical articles as a little less than 100,000, a list of 38,748 entries, with comparatively few repetitions and confined to the preceding century, is sufficiently extensive to include all the articles showing real advances. Even if this high ideal has not been entirely reached, as only about 750 articles appear to have been rejected on account of their lack of sufficient permanent interest to be entered, yet the present volume is extremely valuable and it is hard to see how any live mathematical scholar can afford to be without it.

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* Cf. Günther, Cantor's Vorlesungen über Geschichte der Mathematik, vol. 4 (1908), p. 3.

† Felix Müller, *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 12 (1903), p. 439.

For the greater part of the period covered, the volume under review contains the most complete subject catalogue extant. In fact, the only other work which can be compared with it is the well-known Répertoire bibliographique des sciences mathématiques, which was started with a view to covering the period from 1800 to 1890, but is still far from being complete. Since 1871 the *Jahrbuch über die Fortschritte der Mathematik* has been doing excellent service, and since 1893 the *Revue semestrielle des publications mathématiques* has made it much easier to keep in touch with recent literature, and the publication of separate volumes covering five-year periods has made the *Revue* very much more serviceable. Although such publications are very useful in looking up the total literature on a subject, they can scarcely be compared with a work of such a large scope as that of the volume under consideration.

A commendable feature of the extensive bibliographic undertaking of the Royal Society is that it tends to make it easier to keep in touch with progress in more than one of the large fields of scientific inquiry. If the subject catalogues of the various sciences form parts of the same set and are arranged according to the same general plan, it is very much easier to maintain the community of interests which ought to characterize scientific endeavor and which frequently tends towards mutual helpfulness. Even if this feature is not regarded as being of primary importance, it is far-reaching in its effects and seems to deserve more attention than it has been receiving during the recent period of very rapid development. Even in mathematics itself there is danger of estrangement, as Darboux pointed out in his recent address before the fourth international congress of mathematicians.*

When the volume under review is examined in detail there is considerable room for criticism. The classification is so minute as to demand an unusually large amount of mathematical knowledge on the part of the classifier, and it is not surprising to find that this demand was sometimes higher than his patience or knowledge could meet. Hence it is frequently necessary to look under several closely related headings to find the information that may be desired. The titles of the papers published after 1883 were made from the contents of the papers and not merely from their headings. This was also done as regards the earlier papers when the headings were too indefinite for the minute classification demanded by the general scheme.

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* *Bulletin des Sciences Mathématiques*, vol. 32 (1908), p. 108.

As an instance where the heading is entirely misleading we may cite the papers by Cockle on a species of indeterminate analysis, which appeared in the *Cambridge and Dublin Mathematical Journal* (1851–53) under the title "Method of vanishing groups." Although they have nothing in common with what is now known as group theory, they are classed with this subject in the present volume. Bilenki's note on "permutants" is classed with permutations and substitutions instead of with the papers on matrices.

In the classification, the subjects are arranged under the registration numbers adopted by the International Catalogue, and a copy of Schedule A of this catalogue is made to serve as a table of contents of the volume, with references to the pages where the articles relating to the various subjects are indexed. In order to save repetition in printing, a number of sub-headings not contained in the International Catalogue, have been employed. These are printed in italics so that they can readily be distinguished from the regular headings of the catalogue. The list of periodicals which precedes the regular index includes not only the abbreviations used for each journal and the date when it was started, but also a statement as regards the British libraries in which it may be found and whether the library contains a complete or only an incomplete set.

Notwithstanding the imperfections to which attention has been called, the present work fills a great want which fully justifies the expenditure of the vast amount of labor involved in its production. It will doubtless contribute greatly towards progress in our science by making it much easier to become acquainted with everything that has been done along a certain line. Fortunately the price of the volume is sufficiently low to make it practicable for a large number of scholars to have it in their private libraries and thus use it to develop many thoughts which would otherwise make only a passing impression.

G. A. MILLER.

Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat. By H. S. CARSLAW. London, Macmillan and Company, 1906. 8vo. 17 + 434 pp.

THIS book is an attempt to place the mathematical theory of heat on a rigorous basis. On the one hand there are in English many books dealing with such pure mathematical subjects as the theory of infinite series, the theory of definite integrals, and the theory of functions of a real variable, which give rigorous developments of their subjects, and on the other there are books on applied mathematical subjects which in many cases take as axiomatic many of the pure mathematical theorems on which the applied subject is based. Mr. Carslaw is to be congratulated on the success of his endeavor to deal rigorously with the theory of heat. His book divides naturally into two parts. The first deals with such matters as the theory of series and of definite integrals in general, and in particular with the Fourier series and integrals. The second part consists of an application of this machinery to the standard problems of the conduction of heat.

In the chapters on theory of numbers and series attention should be drawn to the clearness of explanation of Dedekind's axiom and of the sum of a series. The student usually finds so much difficulty in understanding what is meant by the sum of an infinite series, that it is worth while to repeat the statement given by Carslaw and due to Baker.

"When we speak of the sum of an infinite series $u_1(x) + u_2(x) + u_3(x) + \cdots$ it is to be understood (1) that we settle for what value of x we wish the sum of the series; (2) that we insert this value of x in the different terms of the series; (3) that we find the sum $S_n(x)$ of the first n terms of the series; and (4) that we find the value of the limit of this sum as n increases indefinitely, keeping x all the time at the value settled upon.

"With this understanding, the series is said to be convergent for the value x, and to have $f(x)$ for its sum, when, this value of x having first been inserted in the different terms of the series, and any positive quantity ϵ having been chosen as small as we please, there exists a finite integer ν such that $|f(x) - S_n(x)| < \epsilon$ for $n > \nu$."

The author gives a good account of uniform convergence. He states and gives a proof of the statement that uniform convergence implies continuity. This is of course true if the terms of the series are all continuous, and the author makes this assumption. It is clear however that a series may satisfy the condition of uniform convergence and yet have a finite number of terms which are not continuous in the interval considered, provided that each of these terms has a definite value at every point in the interval, and in this case the continuity breaks down. Since the author aims so definitely at completeness, he might have included in this chapter (Series) the definition of continuity.

Chapter IV contains a treatment of the theory of infinite integrals. It is noteworthy that though until a year ago there was no detailed treatment of this subject in English, there have appeared almost simultaneously three books which contain full accounts of it, those of Pierpont, Carslaw, and Bromwich.

Chapters V and VI deal with Fourier's series. The discussion of convergence of these series is admirably illustrated by a series of graphs, which give the successive curves of approximation as one, two, three, etc., of the terms of some particular series are taken. These graphs illustrate very well the way in which discontinuities arise in a series of continuous functions. It is interesting to note that as n increases indefinitely the curve $y = S_n(x)$ does not tend precisely to the discontinuous curve $y = S(x)$ and the vertical straight lines joining the points of discontinuity. The vertical lines in fact extend through and beyond the points of discontinuity. This was first pointed out by Gibbs, and has been generalized for Fourier series by Bôcher.

The second part of the book, after a chapter on the general theory, deals with particular classical problems, such as Fourier's ring, the straight rod, the sphere, etc.

The book is not free from errors which should not have escaped the proof reader. For example, the graph on page 59 for the curve $y = nx/(1 + n^2x^2)$ is stated on the figure itself to be drawn for the two cases $n = 10$, $n = 100$. Beneath the figure these numbers are said to be $n = 5$, $n = 50$, whilst in the text they are said to be 5 and 100. In a second edition it might be well to express differently such phrases as "the rapidity of the convergence becomes slower and slower" (page 55), and "By choosing q large enough we may make this as small as we please for that and all greater values of q" (page 182).

The book contains a number of exercises and concludes with a full bibliography and a fairly good index.

J. EDMUND WRIGHT.

Angewandte Potentialtheorie in elementarer Behandlung. Von E. GRIMSEHL. Sammlung Schubert XXXVIII. Leipzig, G. J. Göschen. Vol. 1, vii + 219 pp., 1905.

FOR those who desire to teach the potential theory for the sake of its applications and to classes of students of limited mathematical advancement, Grimsehl has done a real service in writing this book, and promises to do an additional service in the second volume. Starting with the ideas of force and work, the author develops the potential theories for particles of mass and applies the results to the discussion of the attraction due to spherical shells, spheres, and homologous ellipsoids; the conception of lines and tubes of force and of equipotential surfaces are appropriately treated. There immediately follows a discussion of the determination of the constant of gravitation, of the density of the earth, and of other similar problems. The next section of the book takes up electric potential and fields of force with the theory of electric images. The usual theory of conductors and condensors is given in detail and with great clearness, and considerable attention is paid to the quadrant electrometer. The second volume will set forth the applications of potential theory to magnetism, electromagnetic phenomena, and electric currents. It is certainly a great convenience for any teacher of elementary mathematical physics to have all these matters carefully collected, set in relation to one another, and thoroughly explained from the one unifying principle of potential.

E. B. WILSON.

NOTES.

THE Sixteenth Summer Meeting and Sixth Colloquium of the AMERICAN MATHEMATICAL SOCIETY will be held at Princeton University during the entire week, September 13–18, 1909. The first two days will be devoted to the regular sessions for the presentation of papers. At the Colloquium, which will open on Wednesday morning, the following courses of lectures will be delivered: Professor G. A. BLISS, "Existence theorems in analysis;" Professor J. H. JEANS, "Statistical mechanics;" Professor EDWARD KASNER, "Geometric aspects of dynamics."

The Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January.

Blanks for furnishing necessary information have been sent to the members. Early notice of any changes since the issue of the last Register will greatly facilitate the work of the Secretary. The Register is widely circulated and it is desirable that the information which it contains should be accurate and reliable.

AT the meeting of the London mathematical society held on December 10 the following papers were read: By H. LAMB, "On the propagation of sound waves vertically in the atmosphere;" by E. W. HOBSON, "On Sir William Rowan Hamilton's fluctuating functions," and "On the representation of a function by series of Bessel's functions;" by G. H. HARDY, "Theory of Cauchy's principal values (fourth paper);" by T. STUART, "Solution of a problem of Mersenne's;" by L. J. ROGERS, "Note on a continued fraction equivalent to the remainder after n terms of Taylor's series;" by B. RAM, "Solid angles and potentials of plane discs."

AT the third international congress of philosophy, held at Heidelberg, August 31 to September 5, the following papers on the philosophy of mathematics were read: By P. MANSION, "Gauss versus Kant and non-euclidean geometry;" by M. KUNTZE, "The philosophic significance of Grassmann's Ausdehnungslehre;" by M. M. WINTER, "Relation between intuition and mathematical thought;" by A. REY, "Deduction and experience in scientific methods;" by E. MEYERSON, "Scientific explanations and common sense;" by F. ENRIQUES, "Principle of sufficient reason;" by M. DUFUMIER, "The notion of a positive formal logic." In the section of logic and theory of knowledge there was a symposium on mathematical logic, in which E. MÜLLER, M. G. ITELSON and Mrs. LADD-FRANKLIN participated. The next meeting of the congress will be held at Bologna in the summer of 1911.

THE ninth meeting of the Swiss teachers of mathematics was held at Baden, October 4, under the presidency of Professor H. FEHR, and in affiliation with the Swiss society of gymnasium teachers. The following papers were read: By D. RUEFLI, "Presentation of maxima and minima in the secondary schools;" by C. JACCOTET, "Demonstration of a theorem of Descartes;" by H. FEHR and E. GUBLER, "Report of the fourth international congress of mathematicians;" by H. FEHR, "The international commission on mathematical instruction." The retiring officers were reelected. The next session will be held at Bern, May 22, 1909.

IN the series of monographs on mathematical physics for engineers and students published by Teubner, Leipzig-Berlin, three volumes have already appeared and the following are in the press: Principles of ship construction, by O. ALT; Theory of gases, by A. BYK; Mathematical instruments, by A. GALLE; Theory of potential, by R. GANS; Problems in oscillation, by E. GRÜNEISEN; Vector analysis and its applications to mathematical physics, by W. v. IGNATOWSKY; Tables of functional values with formulas and curves. by E. JAHNKE and F. EMDE; Acoustics, by A. KALÄHNE; Thermoelectricity, by F. KRÜGER; Conformal representation, by L. LEWENT; Introduction to the theory of elasticity, by R. MARCOLONGO; Technical hydromechanics, by R. v. MISES; Principles of the mechanics of alternating currents, by E. ORLICH; Fourier series, by R. ROTHE; Partial differential equations, by R. ROTHE; Electromagnetic oscillations, by R. RÜDENBERG; Bessel functions, by P. SCHAFHEITLIN; Measurements of temperature, by S. VALENTINER; Thermodynamics of mixtures, by E. BOSE; Motors for alternating currents, by J. SUMEO.

PROFESSOR G. DARBOUX, of the University of Paris, has been elected to membership in the academy of sciences of Halle.

PROFESSOR G. H. DARWIN, of the University of Cambridge, has been elected corresponding member of the Prussian academy of sciences.

PROFESSOR G. LORIA, of the University of Genoa, has been elected honorary member of the mathematical society of Amsterdam.

PROFESSOR H. A. SCHWARZ, of the University of Berlin, has been decorated with the order of the cross of the second class.

DR. G. W. HILL has been elected corresponding member of the Bavarian academy of science.

DR. E. SCHMIDT, of the University of Bonn, has accepted a call as professor of mathematics at the University of Zurich.

DR. U. CISOTTI has been appointed docent in rational mechanics at the University of Padua.

DR. M. HELLINGER has been appointed docent in mathematics at the University of Marburg.

PROFESSOR G. HESSENBERG, of the agricultural academy at Bonn-Poppelsdorf, has been appointed docent in mathematics at the University of Bonn.

DR. C. MÜLLER has been appointed docent in mathematics at the University of Göttingen.

DR. H. TIETZE has been appointed docent in mathematics at the University of Vienna.

DR. P. BOUTROUX has been appointed associate in rational mechanics at the University of Poitiers, succeeding Dr. H. LEBESGUE, who was recently promoted to the chair of analysis.

DRS. M. GOURSAT and C. A. LAISANT have been promoted to the full rank of examiners, in analysis and mechanics respectively, at the Ecole polytechnique at Paris. Dr. E. MAILLET succeeds Dr. GOURSAT as associate examiner in analysis.

AT Brown University Drs. N. J. LENNES and R. C. ARCHIBALD have been appointed instructors in mathematics.

NEW PUBLICATIONS.

1. HIGHER MATHEMATICS.

Best (L.). Beweis des Fermat'schen Satzes. Darmstadt, Schlapp, 1908. 8vo. 3 pp. M. 0.30

Boehm (K.). Elliptische Functionen. 1ter Teil. Theorie der elliptischen Funktionen, aus analytischen Ausdrücken, entwickelt. (Sammlung Schubert, XXX). Leipzig, Göschen, 1908. 8vo. 12 + 356 pp. Cloth. M. 8.60

Fischer (P. B.). Determinanten. (Sammlung Göschen, 402). Leipzig, Göschen, 1908. 8vo. 134 pp. M. 0.80

Fisher (I.). A brief introduction to the infinitesimal calculus. Revised edition. New York, Macmillan, 1908. 12mo. 84 pp. Cloth. $0.75

Hardy (G. H.). A course of pure mathematics. Cambridge, University Press, 1908. 8vo. 444 pp. 12 s.

Jahrbuch über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Im Verein mit anderen Mathematikern und unter besonderer Mitwirkung von F. Müller, A. Wangerin, E. Salkowski sowie der Berliner mathematischen Gesellschaft herausgegeben von E. Lampe. Vol. 37, 1906. 1tes Heft. Berlin, Reimer, 1908. 8vo. 484 pp. M. 17.00

Koch (J.). Beweis des grossen Fermat'schen Satzes. Borna, Noke, 1908. 8vo. 34 pp. M. 1.25

Kübler (J.). Beweis des Fermat'schen Satzes, dass die Gleichung $x^n + y^n = z^n$ für $n > 2$ in ganzen Zahlen niemals auflösbar sei. Leipzig, 1908. 8vo. 18 pp. M. 1.00

Lampe (E.). See Jahrbuch.

Müller (F.). See Jahrbuch.

Netto (E.). Gruppen- und Substitutionentheorie. (Sammlung Schubert, LV.) Leipzig, Göschen, 1908. 8vo. 8 + 176 pp. Cloth. M. 5.20

Ohrtmann (C.). See Jahrbuch.

Rudio (F.). Die Elemente der analytischen Geometrie. Mit zahlreichen Uebungsbeispielen. 2ter Teil: Die analytische Geometrie des Raumes. 4te, verbesserte Auflage. Leipzig, Teubner, 1908. 8vo. 10 + 194 pp. M. 3.00

Ruhl (H.). Elementarer Beweis des Fermatschen Satzes. Darmstadt, Müller, 1908. 8vo. 4 pp. M. 0.30

Salkowski (E.). See Jahrbuch.

Sturm (R.). Die Lehre von den geometrischen Verwandtschaften. Vol. II. Die eindeutigen linearen Verwandtschaften zwischen Gebilden zweiter Stufe. (B. G. Teubner's Sammlung Vol. XXVII, 2.) Leipzig, Teubner, 1908. 8vo. 8 + 346 pp. Cloth. M. 16.00

Versuch einer Lösung des grossen Fermatschen Satzes: $a^n + b^n = c^n$. (Diss.) Halle, 1908. 8vo. 14 pp. M. 0.60

VOSS (A.). Ueber das Wesen der Mathematik. Rede. Erweitert und mit Anmerkungen versehen. Leipzig, 1908. 8vo. 98 pp. M. 3.60

WANGERIN (A.). See JAHRBUCH.

II. ELEMENTARY MATHEMATICS.

BEHRENDSEN (O.) und GÖTTING. Lehrbuch der Mathematik nach modernen Grundsätzen. A. Unterstufe. Leipzig, Teubner, 1908. 8vo. 7 + 254 pp. M. 2.80

BOREL (E.). Die Elemente der Mathematik. Deutsche Ausgabe. besorgt von P. Stäckel. Vol. 1: Arithmetik und Algebra. Leipzig, Teubner, 1908. 8vo. 14 + 431 pp. Cloth. M. 8.60

DAVISON (C.). Algebra for secondary schools. New York, Putnam, 1908. 8vo. 623 pp. Cloth. $2.00

DELAISTRE (L.). Cours complet de dessin linéaire gradué et progressif, contenant la géométrie pratique, élémentaire et descriptive. 6e édition. Paris, Gauthier-Villars, 1908. 4to. 8 + 65 pp. Fr. 15.00

DIECKMANN (E.). Leitfaden für den Unterricht in Algebra. Rechenergebnisse zu den Aufgaben, zusammengestellt von O. Vedder. Autographiert von R. Priesterback. Leipzig, Degener, 1908. 8vo. 4 + 24 pp. M. 2.00

DIESENER (H.). Die Stereometrie. Mit vielen praktischen Beispielen und Aufgaben. 3te Auflage. Leipzig, Voigt, 1909. 8vo. 4 + 98 pp. M. 3.00

GENAU (A.) und GRÜNDER (J.). Geometrie fur Lehrerbildungsanstalten. Nach den Lehrplänen von 1901 bearbeitet. 14te Auflage. Leipzig, Reisland, 1908. 8vo. 8 + 263 pp. M. 3.75

GEOMETRÍA Y TRIGONOMETRÍA. Respuestas al cuestionario oficial para el ingreso en las escuelas de ingenieros industriales de Madrid y Barcelona. Barcelona, Bosch, 1908. 99 pp.

GÖTTING. See BEHRENDSEN (O.).

GRÜNDER (J.). See GENAU (A.).

KRIMPHOFF (W.). See SCHWERING (K.).

MEHLER (F. G.). Hauptsätze der Elementar-Mathematik zum Gebrauche an höheren Lehranstalten. Bearbeitet von A. Schulte-Tigges. Ausgabe B. Unterstufe. Planimetrie und Arithmetik. Berlin, Reimer, 1908. 8vo. 9 + 204 pp. M. 2.00

MINET (A.) et PATIN. Cours pratique d'arithmétique, de système métrique et de géométrie. Cours élémentaire, 1re et 2e années. 11e édition, revue et corrigée. Paris, Nathan, 1908. 8vo. 192 pp.

MORAGA SÁNCHEZ (F.). Elementos de trigonometría general y rectilínea. Madrid, Ricardo Rojas, 1908. 120 pp.

MÜLLER (H.). Die Elemente der Stereometrie. Ein Beitrag zur Methode des geometrischen Unterrichts. 4te Auflage. Metz, Scriba, 1908. 8vo. 6 + 88 pp. M. 1.60

PAHL (F.). See SCHULTZE (E.).

PATIN. See MINET (A.).

Potter (W. J.). Concurrent practical and theoretical geometry. London, Ralph, 1908. 8vo. 194 pp. 1s. 6d.

Priesterbach (R.). See Dieckmann (E.).

Romero Rubira (A.). Elementos de algebra. Alicante, Lopez, 1908. 8 + 271 pp.

Ross (P.). Elementary algebra. For the use of higher grade and secondary schools. Part II, with answers. London, Longmans, 1908. 8vo. 240 pp. Cloth. 2s. 6d.

Schmehl (C.). Lehrbuch der Arithmetik und Algebra. 2ter Teil. Ausgaben A und B. Auflösungen. Giessen, Roth, 1908. 8vo. M. 1.20

—— Sammlung von Aufgaben aus der Algebra und algebraischen Analysis im Anschlusse an das Lehrbuch: Die Algebra und algebraische Analysis mit Einschluss einer elementaren Theorie der Determinanten. Für die Prima der realistischen Anstalten. Giessen, Roth, 1908. 8vo. 6 — 136 pp. M. 2.00

Schneider (F.). Zur Methodik der Elementar-Mathematik. Winke für Lehramtskandidaten und jüngere Lehrer. Stuttgart, Grub, 1908. 8vo. 6 + 68 pp. M. 1.40

Schulte-Tigges (A.). See Mehler (F. G.).

Schultze (E.) und Pahl (F.). Mathematische Aufgaben. Ausgabe für Realgymnasien, Oberrealschulen und Realschulen. 2ter Teil. Leipzig, Dürr, 1908. 8vo. 8 + 296 pp. M. 3.60

Schuster (M.). Geometrische Aufgaben und Lehrbuch der Geometrie — Planimetrie, Stereometrie, ebene und sphärische Trigonometrie — nach konstruktiv-analytischer Methode bearbeitet. Ausgabe A: für Vollanstalten. 3ter Teil: Stereometrie. 2te, nach den preussichen Lehrplänen von 1901 umgearbeitete Auflage. Leipzig, Teubner, 1908. 8vo. 6 + 104 pp. M. 1.80

Schwering (K.) und Krimphoff (W.). Ebene Geometrie. Nach den neuen Lehrplänen bearbeitet. 6te Auflage. Freiburg, Herder, 1908. 8vo. 10 + 138 pp. M. 2.20

Sievert (H.). Lehrbuch der Elementar-Geometrie zum Gebrauche an Mittelschulen und beim Selbstunterrichte. Leipzig, Deichert, 1908.
1ter Teil. 1te Abteilung. 3te Auflage. 8vo. 6 + 179 pp. M. 2.40
3ter Teil. 1te Abteilung. 2te Auflage. 8vo. 6 + 95 pp. M. 1.60

Stäckel (P.). See Borel (E.).

Strenger (E.). Mathematische Aufgaben aus den Reifeprüfungen der württembergischen Oberrealschulen. 1ter Teil. Leipzig, Quelle, 1908. 8vo. 80 pp. M. 1.25

Vedder (O.). See Dieckmann (E.).

Wiltz (H.). Mathematische Unterrichtsbriefe. Einfuhrung in die Grundzüge der Elementar-Mathematik durch Selbstunterricht. (Unter Mitwirkung von bewährten Fachgelehrten bearbeitet und herausgegeben.) 6ter Lehrbrief mit 6tem Wiederholungsbrief und Einleitungsbrief. Frankfurt a.M., Wolstein, 1908. 8vo. 29 + 7 + 18 pp. M. 3.00

Wolff (H.). Sätze und Aufgaben der Geometrie für Realanstalten, nach den preussischen Lehrplänen von 1901 zusammengestellt. 1ter Teil. Pensum der Unterstufe (Quarta bis Untersekunda). Leipzig, Teubner, 1908. 8vo. 4 + 98 pp. M. 1.60

III. APPLIED MATHEMATICS.

Anthony (W. A.). Lecture notes on the theory of electrical measurements. 3rd edition, revised by A. Ball. New York, Wiley, 1908. 12mo. 9 + 122 pp. Cloth. $1.00

Ball (A.). See Anthony (W. A.).

Blondin (J.) See Poincaré (H.).

Brown (E. W.). The inequalities in the motion of the moon due to the direct action of the planets; Adams prize essay, University of Cambridge, 1907. New York, Putnam, 1908. 4te. 92 pp. Cloth. $2.00

Dietsch (C.). Leitfaden der darstellenden Geometrie. 5te Auflage. Leipzig, Deichert, 1909. 8vo. 4 + 159 pp. M. 2.20

Ehrig (G.). Geometrie für Baugewerkenschulen und verwandte technische und gewerbliche Lehranstalten mit besonderer Berücksichtigung der praktischen Anwendungen. 1er Teil: Geometrie der Ebene. 2te, vermehrte und verbesserte Auflage. Leipzig, Leineweber, 1909. 8vo. 8 + 138 pp. Cloth. M. 2.40

Hauber (W.). Festigkeitslehre. (Sammlung Göschen, 288). Leipzig, Göschen, 1908. 8vo. 127 pp. M. 0.80

Hotopp (L.). See Keck (W.).

Jacobsen (J. H.). Calcul électrique et mécanique de distributions d'énergie. Paris, l'auteur, 23 Rue Baudin, 1908. 8vo. 40 pp. Fr. 2.25

Johnson (L. J.). Statics by algebraic and graphic methods; intended primarily for students of engineering and architecture. 2nd edition, revised and enlarged. New York, Wiley, 1908. 8vo. 11 + 169 pp. Cloth. $2.00

Keck (W.). Vorträge über Mechanik als Grundlage für das Bau- und Maschinenwesen. 2er Teil: Mechanik elastisch-fester und flüssiger Körper. 3te Auflage, bearbeitet von L. Hotopp. Hannover, Helwing, 1909. 8vo. 12 + 384 pp. M. 13.50

Molesworth (G. L. and H. B.). Pocket book of useful formulae and memoranda for civil, mechanical and electrical engineers. 26th edition, revised and enlarged. London, Spon, 1908. 18mo. Cloth. 5 s.

Perry (J.). Angewandte Mechanik. Ein Lehrbuch für Studierende, die Versuche anstellen und numerische und graphische Beispiele durcharbeiten wollen. Berechtigte deutsche Uebersetzung von R. Schick. Leipzig, Teubner, 1908. 8vo. 6 + 666 pp. Cloth. M 18.00

Schick (R.). See Perry (J.).

Poincaré (H.). Thermodynamique. Rédaction de J. Blondin. 2e édition, revue et corrigée. Paris, Gauthier-Villars, 1908. 8vo. 19 + 459 pp. Fr. 16.00

Prelini (C.). Graphical determination of earth slopes, retaining walls, and dams. New York, Van Nostrand, 1908. 8vo. 9 + 129 pp. Cloth. $2.00

Rust (A.). Ex-meridian altitude, azimuth and star-finding tables. New York, Wiley, 1908. 8vo. 51 + 393 pp. Cloth. $5.00

SÉGUÉLA (R.). Eléments de résistance des matériaux appliquée au béton armé. Paris, Béranger, 1908. 8vo. 130 pp.

SPANGLER (H. W.). Notes on thermodynamics. Part 1. 4th edition, revised. New York, Wiley, 1908. 12mo. 7 + 75 pp. Cloth. $1.00

TIMERDING (H. E.). Geometrie der Kräfte. (B. G. Teubner's Sammlung Vol. I.) Leipzig, Teubner, 1908. 8vo. 12 + 381 pp. Cloth. M. 16.00

WITZ (A.). Thermodynamique à l'usage des ingénieurs. 3e édition. Paris, Gauthier-Villars, 1908. 16mo. 180 pp.

WOODS (R. J.). Strength and elasticity of structural members. 2d edition. New York, Longmans, 1908. 8vo. 11 + 315 pp. Cloth. $3.50

Book No ...
RARY

THE SECOND REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE second regular meeting of the Southwestern Section of the Society was held at the University of Kansas, Lawrence, Kansas, on Saturday, November 28, 1908. About fifty persons, including the following seventeen members of the Society, were present:

Professor W. C. Brenke, Miss Julia T. Colpitts, Professor E. C. Colpitts, Professor E. W. Davis, Dr. Otto Dunkel, Professor E. P. R. Duval, Professor C. C. Engberg, Professor A. B. Frizell, Mr. G. W. Hartwell, Professor E. R. Hedrick, Professor O. D. Kellogg, Mr. W. A. Luby, Professor H. B. Newson, Dr. Mary W. Newson, Professor W. H. Roever, Professor H. E. Slaught, Professor Paul Wernicke.

The morning session was opened at 10 A. M. and the afternoon session at 2 P. M., Professor Newson presiding. The reading of papers was followed by a business meeting, at which Columbia, Mo., was fixed as the next place of meeting, and Professors Davis (chairman), Roever, and Kellogg (secretary) were elected as program committee for the ensuing year.

The following papers were presented:

(1) Professor J. N. VAN DER VRIES: "The steinerians of quartic surfaces."

(2) Professor G. A. MILLER: "On the groups generated by two operators satisfying the condition $s_1s_2 = s_2^{-2}s_1^{-2}$."

(3) Professor E. W. DAVIS: "The imaginary elements of the exponential curve."

(4) Mr. MEYER GABA: "A necessary condition for the Cremona transformation of curves" (preliminary communication).

(5) Professor W. M. ROEVER: "Optical interpretation of some problems in statics."

(6) Professor W. C. BRENKE: "Transformation of series by means of functions admitting a recurrent relation."

(7) Professor A. B. FRIZELL: "Some sets whose cardinal is the second transfinite number."

(8) Professor PAUL WERNICKE: "Note on linkages."

(9) Professor A. B. FRIZELL: "A theorem on operation groups."

(10) Professor E. R. HEDRICK: "Integrals independent of the path in a complex field."

(11) Mr. G. W. HARTWELL: "Fields of force invariant under projective transformations."

(12) Professor PAUL WERNICKE: "Note on curvatures."

Professor Van der Vries and Mr. Gaba were introduced by Professor Newson. Professor Miller being absent, his paper was read by title. This paper appeared in the January BULLETIN. Abstracts of the other papers follow, numbered to correspond with the above list.

1. Professor Van der Vries considered the steinerian of the developable quartic both geometrically and analytically, and finds that it represents all of space. The quartic

$$ax^2s^2 + by^2z^2 + xyzs = 0$$

is examined analytically and the steinerian is found to be each of the four uninodal planes of the surface seven times and another quartic surface having all the singularities of the original quartic. The steinerian of the general quartic surface being of degree 32 is too difficult to admit of analytical methods. Surfaces having particular singularities are considered and the corresponding points of the steinerian located. The paper closes with a few general statements and proofs of properties of hessians and the corresponding properties of the steinerians.

3. Using the method of representing imaginaries which Professor Davis has previously presented to the Society, the periodicity of the exponential function as well as the peculiarity at infinity are interpreted geometrically.

4. Every Cremona transformation can be reduced to a series of quadratic Cremona transformations and collineations. A well-known necessary condition that a curve be transformable into another by a quadratic (and consequently by the general) Cremona transformation is that the genus of the two curves be the same. In Mr. Gaba's paper, a definition of complete rth adjoint, variable rth adjoint, and complement is given. A second necessary condition for the transformability of one curve into another is then derived., viz., that both curves have rth

variable adjoints of the same genus and the same degree of freedom, and that both curves lack the same variable sth adjoints.

5. Consider an apparatus consisting of a smooth wire and two small rings, P_1 and P_2, not on the wire, all rigidly attached to a frame. The wire is in the form of any curve, and on it a small weightless ring R is capable of sliding without friction. A weightless and perfectly flexible string with a weight at one end, after passing through the rings P_2 and R, has its other end attached to P_1. The string slides through the rings P_2 and R without friction. The ring R, under the action of the forces which act upon it, will be in equilibrium at certain points P of the wire. If now the rings P_1 and P_2 be replaced by the eye of an observer and a point source of light respectively, the observer will see images (*i. e.*, actual brilliant points *) of the light at certain points Q of the wire. The first theorem of Professor Roever's paper is that the points P and Q are identical.

Let us now think of a weightless and perfectly flexible string with both of its ends attached at fixed points P_1 and P_2. On the string a small *heavy* ring R is capable of motion without friction. Consider this apparatus as situated in any field of force for which a force function exists. The ring R, under the action of the forces which act upon it, will be in equilibrium at certain points P of the field. If now the points P_1 and P_2 be replaced by the eye of an observer and a point source of light respectively, the observer will see at P an image of the light (*i. e.*, an actual brilliant point) in the equipotential surface which passes through P. If no force function exists, the point P will be a virtual extra brilliant point of the line of force which passes through P. If in particular the points P_1 and P_2 coincide, the apparatus becomes a plumb line. Then the point P is the image which the observer sees of his eye in the equipotential surface which passes through P.

An apparatus was shown which illustrates the first theorem, and an apparatus was described which illustrates the second theorem. In the second apparatus the equipotential surface is actually assumed by the surface of water which reflects light.

6. In Professor Brenke's paper the series $\sum_1^\infty u_n\psi_n(x)$ is transformed by means of an assumed recurrent relation of the

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* For the definitions of the different kinds of brilliant points, see *Transactions Amer. Math. Society*, vol. 9, No. 2, pp. 245–279.

form $\phi_t\psi_n = \alpha_n\psi_{n+r} + \beta_n\psi_{n+s}$, and a theorem on the convergence of the series is stated. Application is made to the cases where $\psi_n(x)$ is one of the functions cos nx, Hermite's polynomial $U_n(x)$, Legendre's polynomial $X_n(x)$, and Bessel's function $J_n(x)$. By specializing the value of x, certain transformations and convergence tests are obtained for the series $\sum_1^\infty u_n$.

7. All permutations of a finite set of symbols may be obtained from any one of them by the enumerative process of filling the first place in all possible ways, then the second, and so on. The example of the continuum proves that this does not hold for infinite sets. Professor Frizell points out that a set whose cardinal is that of all permutations of the series 1 2 3 . . . can be obtained from this series by transposing pairs of consecutive numbers, *e. g.*, 21 43 56 . . . a property which obviously is not shared by finite sets. Transpositions introduced according to the method used by Hardy in building up a set whose cardinal is the second transfinite number $\aleph_1$ furnish an interesting set of sequences with the same cardinal.

8. If by means of non-intersecting lines (links) we connect $2v$ points (vertices) in 3-space, linking each of them to three others, we obtain a linkage with simple vertices. Such linkages are formed by the boundaries of certain maps. The proof of the map color theorem rests on showing that the links of a map's linkage may be marked by 3 indices α, β, γ so that no two bearing the same mark concur in a vertex. It is not possible, as Tait suggests, to apply such marking to the general simple vertex linkage, but it is shown in Professor Wernicke's paper that the conditions under which it can be done, cover the case of the "planar" linkage (that of the plane map).

9. Professor Frizell's second paper establishes the following proposition: Given two rules of combination of which the first is distributive relatively to the second, and a denumerable set of symbols forming a group with respect to the first rule, a semi-group with respect to the second, and a simply ordered class with reference to both, it is possible so to define extensions of the two given rules to the class of fundamental sequences of the given symbols that it shall possess the given properties with regard to the extended rules.

10. Independence of the path of integration is a characteristic property of analytic functions of a complex variable. Professor Hedrick first discusses integrals of the form $I_c = \int_c f'(z)dz$, where $f(z)$ is not analytic, and where $f'(z)$ is the value of the derivative in the direction of the path C. It results that I_c is independent of the path, that $\int_{z_0}^{z} f'(z)dz = f(z) - f(z_0)$, and that the values of $f'(z)$ on the elements of any radial field determine $f(z)$ except for a constant. The properties of $f'(z)$ on such a field are discussed.

In close connection with the preceding stands the discussion of integrals of the form $I_c = \int_c f(z)d\phi(z)$, where $f(z)$ and $\phi(z)$ are any functions of a complex variable. Such integrals, for real functions, have been considered by Kowalewski and by Ingold. Professor Hedrick shows that the integral I_c will be independent of the path, and that the formula for integration by parts holds if $f(z)$ and $\phi(z)$ together satisfy certain symmetrical conditions analogous to the Cauchy-Riemann equations, which turn out to be precisely the Beltrami forms for the surface element computed from $\phi(z)$.

11. A particle moving freely in a plane under the action of a force which depends only on the position of the particle has equations of motion of the form

$$md^2x/dt^2 = \phi(x, y), \quad md^2y/dt^2 = \psi(x, y).$$

The differential equation of the trajectories described by a particle under such conditions has the form

$$y''' = Hy''^2 + Gy'',$$

in which H and G are functions of y', ϕ, ψ, and the first partial derivatives of ϕ and ψ. This type of differential equation is invariant only under the group of collineations. In Mr. Hartwell's paper, the functions ϕ and ψ are found so that this differential equation will be invariant under the several groups of collineations classified by Lie. These functions do not exist in the case of groups of more than five parameters. The integral equations of the trajectories are readily obtained when the group under which the trajectories are invariant contains three or more parameters. The paper will appear in the *Transactions.*

12. For $n = 2, 3$, etc., the determinant

$$\begin{vmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ 1 & x_1' & x_2' & \cdots & x_n' \\ 1 & x_1'' & x_2'' & \cdots & x_n'' \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_1^{(n+1)} & x_2^{(n+1)} & \cdots & x_n^{(n+1)} \end{vmatrix}$$

represents the $n\,!$-fold area, volume, etc., of the figure in n-space determined by $n + 1$ points $(x_1, x_2 \cdots, x_n)$, $(x_1', x_2', \cdots, x_n')$, $\cdots, (x_1^{(n+1)}, x_2^{(n+1)}, \cdots, x_n^{(n+1)})$ and by the (higher) planes connecting every two, every three of them, etc. This figure possesses $\frac{1}{2}(n + 1)n$ edges: $P_{01}, P_{02}, \cdots, P_{n, n+1}$, of lengths p_{01}, $p_{02} \cdots, p_{n, n+1}$, respectively; $(n + 1)n(n - 1)/3\,!$ 2-planes: P_{012}, $P_{013} \cdots, P_{n-1, n, n+1}$; $\cdots$ finally $n + 1$ $(n - 1)$-planes: $P_{0, 1 \ldots, n}$, $P_{0, 1 \ldots, n+1} \cdots, P_{1, 2 \ldots, n+1}$.

Another expression for the same quantity is obtained by repeated multiplication of "bases" with corresponding altitudes:

$$p_{01} \cdot p_{12} \sin(P_{01}, P_{12}) \cdot p_{23} \sin(P_{12}, P_{23}) \sin(P_{012}, P_{123}) \\ \cdot p_{34} \sin(P_{23}, P_{34}) \sin(P_{123}, P_{234}) \sin(P_{0123}, P_{1234}) \cdot \cdots$$

Taking $n + 1$ "consecutive" points on a general curve in n-space, and dividing each factor of this expression by the arc element ds, it becomes $c_1^{n-1} c_2^{n-2} \cdots c_{n-1}$, c_ν denoting the νth curvature, while the determinant, divided by $ds^{\frac{1}{2}(n+1)n}$, easily reduces to

$$\begin{vmatrix} x_1' & x_2' & \cdots & x_n' \\ x_1'' & x_2'' & \cdots & x_n'' \\ \vdots & \vdots & & \vdots \\ x_1^{(n+1)} & x_2^{(n+1)} & \cdots & x_n^{(n+1)} \end{vmatrix},$$

where the upper indices now refer to differentiation with respect to s. Professor Wernicke thus gives a simple and general method for obtaining expressions for the various curvatures of curves in n-space.

O. D. KELLOGG,
Secretary of the Section.

REMARKS CONCERNING THE SECOND VARIATION FOR ISOPERIMETRIC PROBLEMS.

BY PROFESSOR OSKAR BOLZA.

IN his lectures on the calculus of variations (Göttingen, 1904–05), Hilbert has given an elegant modification of Weierstrass's proof of Euler's rule for isoperimetric problems, which reduces the proof to the consideration of an ordinary extremum with a condition. The object of the present note is to show how the same method can also be applied to the *second variation*.

§ 1. *Hilbert's Proof of Euler's Rule.**

We consider the problem of minimizing the integral

$$(1) \qquad J = \int_{x_1}^{x_2} f(x, y, y')dx, \qquad y' = \frac{dy}{dx},$$

with respect to the totality of curves $y = y(x)$ of class C' which join two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$, lie in a certain region $\mathfrak{R}$ of the x, y-plane, and furnish for the integral

$$(2) \qquad K = \int_{x_1}^{x_2} g(x, y, y')dx$$

a given value l. The functions f and g are supposed of class C''' in the domain

$$(x, y) \text{ in } \mathfrak{R}, \qquad -\infty < y' < +\infty.$$

Let

$$\mathfrak{E}_0: \quad y = y(x), \quad x_1 \leqq x \leqq x_2$$

be a solution of the problem which lies in the interior of the region $\mathfrak{R}$, and which is not at the same time an extremal for the integral K. Then we can obtain, according to Weierstrass, a set of admissible variations of $\mathfrak{E}_0$ as follows:

Let $\eta_1(x)$, $\eta_2(x)$ be two arbitrary functions of x of class C' in the interval $(x_1 x_2)$, and vanishing at x_1 and x_2; then the curve

$$y = y(x) + \epsilon_1\eta_1(x) + \epsilon_2\eta_2(x) \equiv Y(x, \epsilon_1, \epsilon_2), \qquad x_1 \lesseqgtr x \lesseqgtr x_2$$

---

*The same proof is given by Kneser, "Euler und die Variationsrechnung," *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Bd. XXV (1907), p. 50.

will be an admissible variation of $\mathfrak{E}_0$ provided that the two constants ϵ_1, ϵ_2 satisfy the relation

$$(3) \qquad K(\epsilon_1, \epsilon_2) \equiv \int_{x_1}^{x_2} g(x, Y, Y')\, dx = l.$$

Hence,* *the function*

$$J(\epsilon_1, \epsilon_2) \equiv \int_{x_1}^{x_2} f(x, Y, Y')dx$$

of the two variables ϵ_1, ϵ_2 *must have, for* $\epsilon_1 = 0$, $\epsilon_2 = 0$, *a minimum restricted by condition* (3).

But if a function $f(x, y)$ has for $x = a$, $y = b$ a (relative) minimum restricted by the condition

$$(4) \qquad \phi(x, y) = 0,$$

and if the two derivatives $\phi_x(a, b)$, $\phi_y(a, b)$ are not both zero, there exists a quantity λ such that

$$(5) \quad f_x(a, b) + \lambda\phi_x(a, b) = 0, \qquad f_y(a, b) + \lambda\phi_y(a, b) = 0,$$

and moreover, if we put

$$F = f + \lambda\phi,$$

the inequality

$$(6) \qquad F_{xx}\phi_y^2 - 2F_{xy}\phi_x\phi_y + F_{yy}\phi_x^2 \Big|_{y=b}^{x=a} \geqq 0$$

must hold.†

Applying these results to the function $J(\epsilon_1, \epsilon_2)$, it follows in the first place that there must exist a constant λ such that the two equations

$$(7) \qquad J_1 + \lambda K_1 = 0, \quad J_2 + \lambda K_2 = 0$$

hold simultaneously, where

$$J_i = \left(\frac{\partial J}{\partial \epsilon_i}\right)_0 = \int_{x_1}^{x_2} (f_y\eta_i + f_{y'}\eta_i')dx,$$

$$K_i = \left(\frac{\partial K}{\partial \epsilon_i}\right)_0 = \int_{x_1}^{x_2} (g_y\eta_i + g_{y'}\eta_i')dx,$$

---

* Here Hilbert's proof branches off from Weierstrass's; the latter proceeds by solving (3) with respect to ϵ_2. Compare for instance my Lectures, p. 203.

† In order to prove these statements, it is only necessary to solve, by means of Dini's theorem on implicit functions, equation (4) with respect to y, $y = \mathfrak{y}(x)$, and to apply the ordinary rule for an extremum of functions of one variable to the compound function $f(x, \mathfrak{y}(x))$. The proof presupposes that f and ϕ are of class C'' in the vicinity of (a, b).

provided, however, that K_1 and K_2 are not both zero. But since $\mathfrak{E}_0$ is not an extremal for the integral K, we can always choose η_2 so that $K_2 \neq 0$.

The function η_2 being so chosen, the second of the equations (7) determines λ and shows that the value of λ is certainly independent of the choice of the function η_1. Hence it follows from the first of the equations (7), according to Du Bois-Reymond's lemma, that $y(x)$ *must satisfy the differential equation*

$$h_y - \frac{d}{dx} h_{y'} = 0, \tag{8}$$

where

$$h = f + \lambda g. \tag{9}$$

§ 2. *Application of Hilbert's Method to the Second Variation.*

We proceed next to apply the second necessary condition (6) to the extremum of the function $J(\epsilon_1, \epsilon_2)$. The condition takes here the form

$$\begin{aligned} K_2^2(J_{11} + \lambda K_{11}) - 2K_1K_2(J_{12} + \lambda K_{12}) \\ + K_1^2(J_{22} + \lambda K_{22}) \geqq 0, \end{aligned} \tag{10}$$

where

$$J_{ij} = \left(\frac{\partial^2 J}{\partial \epsilon_i \partial \epsilon_j}\right)_0, \qquad K_{ij} = \left(\frac{\partial^2 K}{\partial \epsilon_i \partial \epsilon_j}\right)_0.$$

But by an easy computation the inequality (10) reduces to

$$\int_{x_1}^{x_2} (h_{yy}\eta^2 + 2h_{yy'}\eta\eta' + h_{y'y'}\eta'^2)dx \geqq 0, \tag{11}$$

where

$$\eta = K_2\eta_1 - K_1\eta_2. \tag{12}$$

The inequality (11) must hold for all functions η_1, η_2 of class C' which vanish at x_1 and x_2, and it must be remembered that K_1 and K_2 vary with η_1 and η_2 respectively.

From the definition of K_1 and K_2 it follows that

$$\int_{x_1}^{x_2} (g_y\eta + g_{y'}\eta')dx = 0. \tag{13}$$

Conversely, the equation (12) represents the most general function η of class C' which vanishes at x_1 and x_2 and satisfies (13). For let η be any function satisfying these three conditions. Then choose for η_2 any function which satisfies the two first of them and for which $K_2 \neq 0$. Let c be a constant arbitrarily chosen and determine η_1 by the equation

$$\eta = K_2\eta_1 - c\eta_2.$$

Then η_1 is of class C', vanishes at x_1 and x_2, and $c = K_1$, as follows from (13).

Hence we obtain the result: *The inequality* (11) *must be satisfied for all functions η of class C' which vanish at x_1 and x_2 and satisfy the relation (13).*

This is the theorem which we wished to establish by means of Hilbert's method. In order to recognize the advantages of the new proof, let us compare it with the ordinary proof which proceeds as follows:

Suppose we had found — no matter how — a one-parameter set of admissible variations of the curve $\mathfrak{C}_0$,

$$y = \bar{y}(x, \epsilon). \tag{14}$$

Then we must have for this set

$$\delta^2 J \geqq 0. \tag{15}$$

In the discussion of this inequality *two difficulties* arise which have no analogue in the corresponding discussion for the unconditioned problem:

1) The integrand in $\delta^2 J$ contains *the second variations* $\delta^2 y$, $\delta^2 y'$, whereas in the unconditioned problem only the first variations δy, $\delta y'$ occur, owing to the fact that in the latter problem it is sufficient to consider variations of the simplest type

$$\bar{y}(x, \epsilon) = y(x) + \epsilon\eta(x),$$

which is no longer possible in the isoperimetric problem.

This difficulty is removed by eliminating* $\delta^2 y$, $\delta^2 y'$ by means of the equation $\delta^2 K = 0$, the result being

$$\delta^2 J + \lambda\delta^2 K \geqq 0. \tag{1}$$

2) This inequality, which contains only the first variations δy, $\delta y'$, must hold for all functions δy which can be derived from an admissible variation (14) by the δ-process. But the totality of these functions is identical, according to a lemma† due

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* Compare the remarks by Swift, BULLETIN, vol. 14 (1908), p. 373.

† Compare for instance my Lectures, p. 214.

to Weierstrass, with the totality of those functions δy of class C' which vanish at x_1 and x_2 and satisfy the relation $\delta K = 0$.

The proof of this lemma — which is an essential step in the chain of conclusions, and whose omission forms a serious gap in the older theory — constitutes the second difficulty.

Neither of these difficulties occurs in the proof which we have given above.

Freiburg, i. B.,
November 19, 1908.

NOTES ON THE SIMPLEX THEORY OF NUMBERS.

BY PROFESSOR R. D. CARMICHAEL.

(Read before the American Mathematical Society, October 31, 1908.)

I. *Continued Product of the Terms of an Arithmetical Series.*

1. Let a and c be two relatively prime positive integers and form the arithmetical series

$$xa + c, \quad (x = 0, 1, 2, \cdots, n - 1).$$

If we inquire what is the highest power of a prime p contained in the product

$$\prod_{x=0}^{x=n-1} (xa + c), \quad a \not\equiv 0 \pmod{p},$$

we shall find that the general result takes an interesting form. The solution of the problem may be effected in the following manner:

Evidently there exists some number x such that $xa + c$ is divisible by p. Let i be the smallest value of x for which this division is possible, and let c_1 be the quotient thus obtained. Using the notation

$$H\{y\} \tag{1}$$

to represent the index of the highest power of p contained in y, we will show that

$$H\left\{\prod_{x=0}^{x=n-1} (xa + c)\right\} = H\left\{\prod_{x=0}^{x=e_1} (xa + c_1)\right\} + e_1 + 1, \tag{2}$$

where

$$e_1 = \left[\frac{n-1-i_1}{p}\right]$$

is the largest integer not greater than $(n-1-i_1)/p$. In order to prove (2) we have only to notice that in the product of its first member only factors of the form

$$(mp+i_1)a+c$$

contain p and that the quotient of the division is always of the form

$$ma+c_1,$$

and that e_1 is the highest possible value of m. Performing the same operation on the H-function of the second member and continuing the process, we should finally arrive at a number which is simply the index of the required power of p.

In order to write this result in a convenient form let us define a suitable notation. Let i_r be the least integer such that $i_r a + c_{r-1}$ contains p and let c_r be the quotient of this division. For uniformity, set $c = c_0$ and $n-1=e_0$. Further, let e_r be defined by

$$\left[\frac{e_{r-1}-i_r}{p}\right]=e_r. \tag{3}$$

Also let t be the first subscript for which

$$c_t(a+c_t)(2a+c_t)\cdots(e_t a+c_t)$$

does not contain the factor p. Then the preceding result may be written thus

$$H\left\{\prod_{x=0}^{x=n-1}(xa+c_0)\right\}=\sum_{r=1}^{r=t-1}(e_r+1). \tag{4}$$

Since $0 \leqq i_r \leqq p-1$, as is evident from the definition of i_r, we may deduce from (3) the following inequalities:

$$\left[\frac{e_{r-1}-(p-1)}{p}\right] \leqq e_r \leqq \left[\frac{e_{r-1}}{p}\right].$$

Hence

$$\left[\frac{e_{r-1}+1}{p}\right] \leqq e_r+1 \leqq \left[\frac{e_{r-1}+p}{p}\right]. \tag{5}$$

This gives

$$\left[\frac{n}{p}\right] \leqq e_1 + 1 \leqq \left[\frac{n-1}{p}\right] + 1,$$
$$\left[\frac{n}{p^2}\right] \leqq e_2 + 1 \leqq \left[\frac{n-1}{p^2}\right] + 1,$$
$$\left[\frac{n}{p^3}\right] \leqq e_3 + 1 \leqq \left[\frac{n-1}{p^3}\right] + 1,$$
$$\cdots\cdots\cdots\cdots$$

Taking the sum of these inequalities, we have by (4)

$$(6)\qquad \begin{aligned} \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots &\leqq H\left\{\prod_{x=0}^{x=n-1}(xa + c_0)\right\} \\ &\leqq \left[\frac{n-1}{p}\right] + \left[\frac{n-1}{p^2}\right] + \cdots + R(n-1), \end{aligned}$$

where $R(n-1)$ is the index of the highest power of p not greater than $n-1$.

This result takes different forms according as n is or is not a power of p. If n is a power of p, we have evidently

$$(7)\qquad \left[\frac{n}{p^\alpha}\right] = \left[\frac{n-1}{p^\alpha}\right] + 1$$

for every p^α equal to or less than n. Remembering that when $n = p^h$

$$\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \cdots = \frac{p^h - 1}{p - 1},$$

and using equation (7) in connection with inequality (6), we have

$$(8)\qquad H\left\{\prod_{x=0}^{x=n-1}(xa + c_0)\right\} = \frac{n-1}{p-1}, \quad n = p^h.$$

When n is not a power of p, it is evident that

$$(9)\qquad \left[\frac{n}{p^\alpha}\right] = \left[\frac{n-1}{p^\alpha}\right].$$

Suppose now that

$$(10)\qquad n = \delta_h p^h + \delta_{h-1} p^{h-1} + \cdots + \delta_1 p + \delta_0, \quad \delta_h \neq 0,$$

and at least one other δ is not zero. Employing (9) and the well-known formula

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots=\frac{n-(\delta_k+\delta_{k-1}+\cdots+\delta_1+\delta_0)}{p-1},$$

we may write (6) as follows:

$$(11)\qquad \begin{aligned}\frac{n-(\delta_k+\cdots+\delta_1+\delta_0)}{p-1}&\leqq H\left\{\prod_{x=0}^{x=n-1}(xa+c_0)\right\}\\&\leqq h+\frac{n-(\delta_k+\cdots+\delta_1+\delta_0)}{p-1}.\end{aligned}$$

The inequalities in (11) confine the value of H in narrow limits which are easily calculated.

2. In the series $xa+c$, it may happen that the first x for which $xa+c$ is divisible by p will give c as the quotient of this division. Then in the preceding discussion all the c's are equal; and then also all the i's. Dropping subscripts from i and c and making repeated use of equation (3), we have

$$e_1=\left[\frac{n-1-i}{p}\right],$$

$$e_2=\left[\frac{e_1-i}{p}\right]=\left[\frac{e_1p-ip}{p^2}\right]=\left[\frac{n-1-i-ip}{p^2}\right],$$

$$e_3=\left[\frac{e_2-1}{p}\right]=\left[\frac{e_2p^2-ip^2}{p^3}\right]=\left[\frac{n-1-i-ip-ip^2}{p^3}\right],$$

. . . .

If we add one to each member of each of these equations and take the sum of the results; then further, if we replace the resulting first member by its value taken from (4), we have

$$(12)\qquad \begin{aligned}H\left\{\prod_{x=0}^{x=n-1}(xa+c)\right\}&=\left[\frac{n-1-i+p}{p}\right]\\&+\left[\frac{n-1-i-ip+p^2}{p^2}\right]+\left[\frac{n-1-i-ip-ip^2+p^3}{p^3}\right]+\cdots.\end{aligned}$$

3. If $a=c=1$, equation (12) takes a very simple form. For this case $i=p-1$. The result is the well-known theorem that the highest power of p contained in $n!$ is that whose index is

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots=\frac{n-(s_k+\cdots+s_1+s_0)}{p-1},$$

where

$$n = s_\lambda p^\lambda + s_{\lambda-1}p^{\lambda-1} + \cdots + s_1 p + s_0.$$

4. If $a = 2$ and $c = 1$, equation (12) takes a special form of considerable interest. The terms of $xa + c$ are the natural odd numbers in order, and p is an odd prime. It is evident that $i = \frac{1}{2}(p - 1)$. Therefore

$$\left[\frac{n - 1 - i - ip \cdots - ip^{\beta-1} + p^\beta}{p^\beta}\right]$$
$$= \left[\frac{2n-2-2i-2ip-\cdots-2ip^{\beta-1}+2p^\beta}{2p^\beta}\right] = \left[\frac{2n-1+p^\beta}{2p^\beta}\right].$$

Then (12) becomes

$$H\{1\cdot 3\cdot 5\cdots(2n-1)\} = \left[\frac{2n-1+p}{2p}\right]$$
$$+ \left[\frac{2n-1+p^2}{2p^2}\right] + \left[\frac{2n-1+p^3}{p^3}\right] + \cdots.$$

II. *An Extension of Fermat's Theorem.*

It will be shown that the congruence

$$x^{\phi(n)} \equiv 1 (\mathrm{mod}\ n),$$

where $\phi(n)$ is Euler's ϕ-function of n, is still true when the modulus is a multiple of n formed in a definite way, x being prime to the new modulus.

It has been shown * that $\phi(z) = a$ has always more than one solution. If z_1 and z_2 are two roots of $\phi(z) = a$, then z_1 and z_2 must each have a factor not common to the two except when one is an odd number and the other is twice that odd number; and hence, except in this case, their lowest common multiple is greater than either of them. Now if $z_1, z_2, \cdots, z_i$ are all the roots of $\phi(z) = a$, we have by Fermat's theorem the congruences

$$x^a \equiv 1(\mathrm{mod}\ z_1),\ x^a \equiv 1(\mathrm{mod}\ z_2),\ \cdots,\ x^a \equiv 1(\mathrm{mod}\ z_i),$$

where in each case x is prime to the modulus involved. Now if L is the lowest common multiple of $z_1, z_2, \cdots, z_i$ and x is prime to L, we have

(1) $$x^a \equiv 1(\mathrm{mod}\ L),$$

where L is greater than any number whose totient is a except when the equation $\phi(z) = a$ has only the two solutions $z = L$, $z = \frac{1}{2}L$. Hence,

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* Carmichael, BULLETIN, vol. 13, p. 241.

THEOREM. *Except when n and $\frac{1}{2}n$ are the only numbers whose totient is the same as that of n, the congruence $x^{\phi(n)} \equiv 1$ holds for a modulus which is some multiple of n.*

A working method for finding such a modulus is the following:

Set $\phi(n) = a$, for convenience. Separate a into its prime factors and find the highest power of each prime p contained in a such that $\phi(p^a)$ is equal to or is a factor of a. Suppose that the following primes are found: $p_1^{a_1}, p_2^{a_2}, \cdots, p_j^{a_j}$. Then write out all the divisors of a and take every prime q such that $q - 1$ is equal to any one of these divisors, but $q \neq$ any p; and say we have $q_1, q_2, \cdots, q_k$. Then set

(2) $$M = p_1^{a_1} p_2^{a_2} \cdots p_j^{a_j} q_1 q_2 \cdots q_k.$$

Then evidently

(3) $$X^a \equiv 1 (\text{mod } M),$$

when X is prime to M. (It should be noticed that M may be a multiple of L in congruence (1).)

As thus defined, M is a definite function of a; say $M = M(a)$. For every odd value of a, except $a = 1$, we have $M(a) = 1$, as the reader may readily verify. Some even values of a give also $M(a) = 1$. There follows a table giving the value of $M(a)$ for each a for which $M \neq 1$ up to $a = 150$.

a	$M(a)$	a	$M(a)$	a	$M(a)$
2	12	48	2 227 680	104	12 720
4	120	52	6 360	106	1 284
6	252	54	43 092	108	22 265 704 680
8	240	56	6 960	110	33 396
10	132	58	708	112	26 740 320
12	32 760	60	3 407 203 800	116	7 080
16	8 160	64	32 640	120	279 390 711 600
18	14 364	66	388 332	126	549 092 628
20	6 600	70	9 372	128	65 280
22	276	72	10 087 262 640	130	17 292
24	65 520	78	948	132	50 483 160
28	3 480	80	18 400 800	136	10 960
30	85 932	82	996	138	1 646 316
32	16 320	84	285 962 040	140	13 539 400
36	69 090 840	88	491 280	144	342 966 929 760
40	108 240	92	5 640	148	17 830
42	75 852	96	432 169 920	150	12 975 732
44	2 760	100	3 333 000		
46	564	102	25 956		

III. *The Solutions of $\phi(z) = a$.*

It is desirable to have a general method for finding all the solutions of

$$\phi(z) = a$$

for any given a. The method used in Note II for finding M in congruence (1) is suggestive, and we may formulate a rule thus:

Find M as in Note II. Evidently, *the solutions of $\phi(z) = a$ will all be factors of M. Then examine all the factors of M and retain each one whose totient is a.*

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THE SOLUTION OF BOUNDARY PROBLEMS OF LINEAR DIFFERENTIAL EQUATIONS OF ODD ORDER.

BY PROFESSOR W. D. A. WESTFALL.

E. SCHMIDT[1] has studied the set of linear integral equations with non-symmetric matrix

$$(1)\quad \phi_i(s) = \lambda_i \int_a^b K(s, t)\psi_i(t)dt, \qquad \psi_i(s) = \lambda_i \int_a^b K(t, s)\phi_i(t)dt,$$

and has shown that, if there can be found for a function $f(x)$ a continuous function $h(x)$, such that

$$(2)\qquad f(x) = \int_a^b K(x, t)\, h(t)dt,$$

then

$$(3)\qquad f(x) = \sum_i \frac{\phi_i(x)}{\lambda_i} \int_a^b h(t)\, \psi_i(t)dt,$$

where ϕ_i runs over a complete set of solutions of (1) which have been normalized and orthogonalized, *i. e.*,

$$(4)\qquad \int_a^b \phi_i \psi_j dx = \left\{ \begin{array}{ll} 1, & i = j, \\ 0, & i \neq j. \end{array} \right\}$$

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[*] *Math. Annalen*, vol. 63, p. 459.

A connection can be established with the boundary problem of differential equations as follows:

If the linear differential expression of odd order $L(y)$ is equal to its adjoint with negative sign, it satisfies the relation

$$(5)\quad \int_a^b [zL(y) + yL(z)]\,dx = [P(y, z)]_a^b + [p_0(zy^{(n-1)} + yz^{(n-1)})]_a^b$$

$$(6)\qquad L(y) = p_0 y^{(n)} + p_1 y^{(n-1)} + \cdots + p_{n-1} y',$$

provided the p, y, and z satisfy certain continuity restrictions. $P(y, z)$ contains no derivatives of y or z of order higher than the $(n-2)$nd.

It is known * that the Green's function of $L(y)$ is skew-symmetric,

$$(7)\qquad G(s, t) + G(t, s) = 0,$$

and that no characteristic solution of $L(y) + \lambda_k y = 0$ exists for a real value of λ. Here $p_0, p_1, \cdots, p_{n-1}$ are real functions of the real variable x.

Let $\lambda_k = l_k + in_k$ be a complex value of λ for which there exists the characteristic solution

$$u_k(x) = \phi_k(x) + i\psi_k(x)$$

of $L(y) + \lambda_k y = 0$. Then $\bar{u}_k = \phi_k(x) - i\psi_k(x)$ is a characteristic solution of $L(y) + \bar{\lambda}_k y = 0$, $\bar{\lambda}_k = l_k - in_k$. Substitute $u = u_k$, $v = \bar{u}_k$ in (5).

$$2l_k \int_a^b u_k(x)\bar{u}_k(x)\,dx = 0.$$

Hence $l_k = 0$. Since u_k is a solution of $L(y) + in_k y = 0$, we have

$$L(\phi_k + i\psi_k) + in_k(\phi_k + i\psi_k) = 0,$$

or

$$(8)\qquad L(\phi_k) - n_k\psi_k = 0, \qquad L(\psi_k) + n_k\phi_k = 0.$$

Since u_k is a characteristic solution, ϕ_k and ψ_k constitute a set of characteristic solutions of equations (8). The relation (5) shows us that these satisfy with the Green's function $G(x, t)$ the equations

$$\phi_k(x) = n_k \int_a^b G(t, x)\psi_k(t)\,dt, \quad \psi_k(x) = -n_k \int_a^b G(t, x)\phi_k(t)\,dt.$$

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* The author's dissertation, Göttingen, 1905.

Or from (7)

$$(9)\quad \phi_k(x) = n_k \int_a^b G(t,x)\,\psi_k(t)\,dt, \qquad \psi_k(x) = n_k \int_a^b G(x,t)\,\phi_k(t)\,dt.$$

Moreover the solutions of equations (9), considered as integral equations with the known matrix $G(x, t)$, give a set of characteristic solutions of (8). This establishes the relation between Schmidt's pair of integral equations and the linear differential equation.

If $f(x)$ is continuous with its first n derivatives and satisfies the boundary conditions satisfied by the Green's function, equation (2) is solved by differentiation

$$L(f) = -h(x).$$

Hence there are an infinite number of pairs of solutions of (8), and an infinite number of characteristic solutions of $L(y) + in_k y = 0$.

COLUMBIA, MO.,
November 30, 1908.

A CLASS OF FUNCTIONS HAVING A PECULIAR DISCONTINUITY.

BY PROFESSOR W. D. A. WESTFALL.

CONSIDER all functions discontinuous for all rational values of the independent variable, and continuous and equal to zero for all irrational values. They are of the form

$$f\left(\frac{p}{q}\right) \neq 0, \quad p \text{ and } q \text{ prime to each other,}$$

$$(1)\qquad f(\alpha) = 0, \quad \text{for } \alpha \text{ irrational, with the condition that}$$

$$\lim_{q=\infty} f\left(\frac{p}{q}\right) = 0.$$

The following are examples of such functions:

$$(2) \qquad \phi_n\left(\frac{p}{q}\right) = \frac{1}{q^n}, \quad \phi_n(1) = 1, \quad \phi_n(\alpha) = 0,$$

$$(3)^* \qquad \phi\left(\frac{p}{q}\right) = \frac{1}{q\,!}, \quad \phi(1) = 1, \quad \phi(\alpha) = 0,$$

$$(4) \qquad \psi\left(\frac{p}{q}\right) = \frac{1}{q^q}, \quad \psi(1) = 1, \quad \psi(\alpha) = 0.$$

Liouville has shown † that if α is an algebraic irrationality of the nth order

$$\left|\frac{p}{q} - \alpha\right| > \frac{A}{q^n},$$

where A is independent of q. Hence ϕ_n has a zero derivative at all algebraic irrationalities of order less than n, and ϕ and ψ at all algebraic irrationalities.

It will now be shown that it is impossible for a function of type (1) to have a derivative for every irrational value of the independent variable. Let $a_1, a_2, a_3, \cdots$ be an infinite sequence of rational numbers such that

$$|a_n - a_{n-1}| < \frac{1}{|f(a_n)|} \quad \text{and} \quad < \left|\frac{1}{q^q}\right|,$$

if $a_n = p/q$. Then this defines a transcendental number α at which the difference quotient of $f(x)$ taken over the sequence a_n is greater than $\frac{1}{2}$. Hence a derivative cannot exist, since it is evidently zero if it exists. The definition of this point shows that such a point exists in every interval.

The above holds equally well for a function discontinuous at all rational points, and continuous at all irrational points in such a way that it coincides in these points with a function having a derivative throughout.

COLUMBIA, MO.,
December 12, 1908.

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* An example first used by Professor Osgood in his lectures.

† *Comptes Rendus* for 1844.

ON CERTAIN DETERMINANTS CONNECTED WITH A PROBLEM IN CELESTIAL MECHANICS.

BY MR. H. E. BUCHANAN.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1909.)

1. THE solutions of the n-body problem, in which all are in the same straight line and moving in circles, are defined by the the following set of equations :*

$$
(1)\qquad
\begin{array}{l}
0+\frac{m_2}{r_{12}^2}+\frac{m_3}{r_{13}^2}+\cdots+\frac{m_n}{r_{1n}^2}=-\omega^2x_1,\\[2ex]
-\frac{m_1}{r_{12}^2}+0+\frac{m_3}{r_{23}^2}+\cdots+\frac{m_n}{r_{2n}^2}=-\omega^2x_2,\\[2ex]
-\frac{m_1}{r_{13}^2}-\frac{m_n}{r_{23}^2}+0+\cdots+\frac{m_n}{r_{3n}^2}=-\omega^2x_3,\\
\quad\vdots\\
-\frac{m_1}{r_{1n}^2}-\frac{m_2}{r_{2n}^2}-\qquad\cdots+0=-\omega^2x_n,
\end{array}
$$

where the m_i stand for the masses and the x_i are their respective distances from the center of gravity of the system.

In his lectures Professor Moulton has considered these equations in two ways :

(1) Given the masses to find the distances.

(2) Given the distances to find the masses.

This paper is concerned with the second problem and its purpose is to answer two questions which he raised.

2. In (2) there is an unique determination of the masses if J, the determinant of the left members, never vanishes. J is a skew symmetric determinant, whose properties are quite different according as its order is even or odd. When n is even it is a perfect square,† and it will be shown that in this case J

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* See Moulton's Celestial Mechanics, p. 213, for the three-body problem. The notation is here chosen so that $x_1 < x_2 < \cdots < x^n$.

† Cayley, Coll. Papers, 1, p. 412.

never vanishes for any selection of r_{ij}. It is only necessary to consider the Pfaffian

$$(2)\qquad J^{\frac{1}{2}} = \left| \begin{array}{ccccc} \frac{1}{r_{12}^2} & \frac{1}{r_{13}^2} & \frac{1}{r_{14}^2} & \cdots & \frac{1}{r_{1n}^2} \\ & \frac{1}{r_{23}^2} & \frac{1}{r_{24}^2} & \cdots & \frac{1}{r_{2n}^2} \\ & & \frac{1}{r_{34}^2} & \cdots & \frac{1}{r_{3n}^2} \\ & & & & \vdots \\ & & & & \frac{1}{r_{n-1,n}^2} \end{array} \right| .$$

Let a definite choice be made for all the x_i from x_1 to x_{n-1} inclusive. Let x_n vary between the limits x_{n-1} and $+\infty$. x_n can be written $x_n = x_{n-1} + x$, where x varies from 0 to $+\infty$. The Pfaffian (2) then becomes a function of x whose properties we propose to study:

$$P(x) = \left| \begin{array}{cccc} \frac{1}{r_{12}^2} & \frac{1}{r_{13}^2} & \cdots & \frac{1}{(x_{n-1} + x - x_1)^2} \\ & \frac{1}{r_{23}^2} & \cdots & \frac{1}{(x_{n-1} + x - x_2)^2} \\ & & & \vdots \\ & & & \frac{1}{x^2} \end{array} \right| .$$

One can verify that $P(x)$ has the following properties:

(*a*) The terms of the last column are arranged in increasing order, $1/x^2$ being the largest.

(*b*) Differentiation as to x serves only to multiply $P(x)$ by a constant factor and to increase the power of the denominators in the last column.

(*c*) $P(x)$, as well as all its derivatives of finite order, remains finite and continuous for $0 < x < \infty$.

(*d*) $P(x)$ and all its derivatives vanish for $x = \infty$.

By Rolle's theorem and (*c*), (*d*), a necessary condition that $P(x)$ vanish for $0 < x < \infty$ is that $P'(x)$ vanish in the same interval. A necessary condition that $P'(x)$ vanish is that $P''(x)$ vanish, and so on; a necessary condition that $P^{(n-1)}(x)$ vanish is that $P^{(n)}(x)$ vanish. On the other hand if $P'(x)$ never vanishes then $P(x)$ does not, and if $P''(x)$ never vanishes $P'(x)$ does not, finally, if $P^{(n)}(x)$ never vanishes $P^{(n-1)}(x)$ does not. From the properties (a) and (b) it follows that for n sufficiently great and $0 < x < \infty$, $P^{(n)}(x)$ never is zero since the term coming from $1/x^2$ will be greater than the sum of all the other terms unless its coefficient vanishes. This coefficient is the Pfaffian of order $n-2$, and we can apply the same method to it as to $P(x)$, so that it does not vanish unless the minor of order $n-4$ vanishes, etc. But the minor of order 2 does not vanish, therefore by induction none of the minors vanishes and the theorem is proven.

3. When n is odd the determinant of the left members of (1) is identically zero.* Then we consider † the first $n-1$ equations. Transpose the terms containing m_n and solve for $m_1, \cdots, m_{n-1}$. This is uniquely possible by § 2. Substituting these solutions in the last equation and arranging, there results

$$F(x_1, \cdots, x_n)\omega^2 + \phi(x_1, \cdots, x_n)m_n = 0.$$

$\phi(x_1, \cdots, x_n)$ is the determinant of the left members and is identically zero since n is odd. In order that there be a solution at all the following condition must be satisfied

$$F(x_1, \cdots, x_n) = 0.$$

m_n can then be taken arbitrarily and the remaining m_j are uniquely determined. Not only so, but it will be shown that after m_n and all of the x's except one has been chosen arbitrarily then that one is uniquely determined by the equation

$$F(x_1, \cdots, x_n) = 0.$$

4. One easily finds

$$F = \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ & \dfrac{1}{r_{12}^2} & \cdots & \dfrac{1}{r_{1n}^2} \\ & & & \vdots \\ & & & \dfrac{1}{r_{n-1,n}^2} \end{vmatrix} = 0.$$

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* Cayley, Coll. Papers, I, p. 412.

† According to Moulton's Lectures.

Choose $x_2, \cdots, x_n$ arbitrary constants, while x_1 varies between the limits $-\infty$ and x_2. It is easily seen that

$$F(-\infty) = \infty, \quad F(x_2) = \infty,$$

the sign being undetermined. Then if the dF/dx_1 never vanishes F must vanish once and but once in the interval $-\infty < x_1 < x_2$. x_1 occurs in the second frame line; therefore on differentiating we have

$$\frac{dF}{dx_1} = \left|\begin{array}{cccc} 1 & x_2 & \cdots & x_n \\ & \dfrac{2}{r_{12}^3} & \cdots & \dfrac{2}{r_{1n}^3} \\ & & & \vdots \\ & & & \dfrac{1}{r_{n-1,n}^3} \end{array}\right| .$$

Now make a definite choice for x_1, say $x_1^{(0)}$, such that

$$-\infty < x_1^{(0)} < x_2;$$

then consider the Pfaffian

$$P_1(\mu) = \left|\begin{array}{cccc} \dfrac{\frac{1}{2}r_{12}^3}{(\frac{1}{2}r_{12} - \mu)^3} & x_2 & \cdots & x_n \\ & \dfrac{2}{(r_{12} - \mu)^3} & \cdots & \dfrac{2}{(r_{1n} - \mu)^3} \\ & & & \vdots \\ & & & \dfrac{1}{r_{n-1,n}^3} \end{array}\right| .$$

Let μ vary from $-\infty$ to $\frac{1}{2}(x_2 - x_1^{(0)}) = \frac{1}{2}r_{12}$; then in this interval $P_1(\mu)$ has the properties :

(*a*) The denominators of the second frame line are arranged in increasing order, $\dfrac{\frac{1}{2}r_{12}^3}{(\frac{1}{2}r_{12} - \mu)^3}$ being the smallest.

(*b*) Differentiation as to μ multiplies P_1 by a constant factor and increases the power of these denominators.

(*c*) $$P_1(0) = \frac{dF}{dx_1}\bigg|^{x=x_1^{(0)}},$$

and P_1 and all its derivatives vanish for $\mu = -\infty$ but remain finite and continuous for $-\infty < \mu < \frac{1}{2}r_{12}$; and finally $P_1(\frac{1}{2}r_{12}) = \infty$. Reasoning then as in §2 we conclude that $P_1(\mu)$ never vanishes for any value, $-\infty < \mu < \frac{1}{2}r_{12}$; in particular it does not vanish when μ has the value 0, that is, by c,

$$\frac{dF}{dx_1}\bigg|^{x=x_1^{(0)}} \neq 0.$$

Since the above proof does not depend on the value assigned to x_1 it follows that dF/dx_1 is never zero for any $-\infty < x_1 < x_2$ and therefore F vanishes once and but once in that interval.

5. In order to complete the theorem one must show that when x_i varies continuously between x_{i-1} and x_{i+1} F vanishes uniquely. The course of reasoning is the same as that above and we have to show that dF/dx_i never vanishes for $r_{i-1} < x_i < x_{i+1}$. x_i occurs in the ith frame line only. Again put $x_i = x_i^{(0)}$ a fixed constant such that $x_{i-1} < x_i^0 < x_{i+1}$ and consider the Pfaffian

$$P_i(\mu) = \left| \begin{matrix} x_1 \cdots & q(\mu) & \cdots & x_n \\ \cdots & q(\mu)\dfrac{2}{r_{1i}^3} & \cdots & \dfrac{1}{r_{in}^2} \\ & \vdots & & \\ & q(\mu)\dfrac{2}{r^3_{i-1,i}} & \cdots & \dfrac{1}{r^2_{i-1,n}} \\ & \dfrac{2}{(r_{i,i+1}-\mu)^3} & \cdots & \dfrac{2}{(r_{in}-\mu)^3} \end{matrix} \right|$$

$$q(\mu) = \tfrac{1}{2}(r^3_{i,i+1})/[\tfrac{1}{2}(r_{i,i+1}) - \mu]^3.$$

It can be shown that the coefficient of $q(\mu)$ in $P_i(\mu)$ does not vanish; then the properties (*a*) (*b*) (*c*) above hold, and the proof is completed as before.

CHICAGO, ILL.,
May, 1908.

SYLVESTER'S MATHEMATICAL PAPERS.

The Collected Mathematical Papers of James Joseph Sylvester. (Edited by H. F. BAKER.) Volume I, xii + 650 pp. Volume II, xvi + 731 pp. Cambridge, University Press, 1904, 1908.

PRIOR to 1877 there had been made in this country a few mathematical investigations of permanent value, in particular that by Benjamin Peirce on linear associative algebras. But such investigations had been done in isolation, and independently, in a measure, of contemporary European activity. These early writers were prophets, as it were, preparing the way for an era of wide-spread activity and interest in mathematical research. Chief among the forces which inaugurated this era was the advent of Sylvester at the opening of John Hopkins University in 1877, and his founding of the *American Journal of Mathematics* the following year. His high ideals for the search after truth, and his unbounded enthusiasm for mathematical science not only proved a powerful stimulus for his colleagues and students, but reached out beyond his immediate surroundings to the country at large.

Sylvester's work in this period, however, does not belong properly to the present review, since the first two volumes of his papers relate to the years 1837–1873. There occur 178 titles, including a few reports or abstracts of communications to societies. These latter titles appear to have been omitted from the Royal Society index, which shows 150 titles down to the year 1873. To judge from the later titles, including eighty-one for the next decade, there will probably be two additional volumes.

Sylvester's investigations related chiefly to algebra, including in that term the theory of invariants, matrices, theory of equations, etc., and the related domains of number theory, differential invariants, etc. However, there are various papers on astronomy, mechanics and physics, while a later volume will include his paper, "An application of the new atomic theory to the graphical representation of the invariants and covariants of binary quantics." His earliest paper related to Fresnel's optical theory of crystals; the next two and a paper of 1850 relate to motion and rest of fluids and rigid bodies. In

1856, there is "A trifle on projectiles" and a letter on Professor Galbraith's construction for the range of projectiles. In 1860 he wrote on the pressure of "mathematical" earth, the "essential quality which differentiates it from actual vulgar earth being continuity."

In the *Philosophical Magazine* for 1866, he has a twenty-four page article entitled "Astronomical prolusions; commencing with an instantaneous proof of Lambert's and Euler's theorems, and modulating through a construction of the orbit of a heavenly body from two heliocentric distances, the subtended chord, and the periodic time, and the focal theory of cartesian ovals, into a discussion of motion in a circle and its relation to planetary motion." After quoting from Lagrange: "His (Lambert's) theorem merits the especial notice of mathematicians, both on its own account, and because it appears difficult to arrive at by algebraical processes; so that it may be ranked among the small number of those in which geometrical seems to have the advantage over algebraical analysis," Sylvester makes some comments which might be called heresy on the part of the future Savilian professor of geometry at Oxford (in succession to the famous number-theorist Henry Smith), viz., "In the nature of things such advantage can never be otherwise than temporary. Geometry may sometimes appear to take the lead over analysis, but in fact precedes it only as a servant goes before his master to clear the path and light him on his way. The interval between the two is as wide as between empiricism and science, as between the understanding and the reason, or as between the finite and the infinite."

In 1866 there is a sequel to the Astronomical prolusions with an equally ambitious title, and later on a supplement to the sequel. In the same year there is a rather extensive paper on the motion of a rigid body acted on by no external forces, and a shorter paper with a similar title. Soon afterwards, Sylvester became much interested in linkages and the general question of the conversion of motion. His lecture before the Royal Institution on January 23, 1874, "On recent discoveries in mechanical conversion of motion," brought to general notice the various possibilities of Peaucellier's linkage and inspired the well-known work of Hart and Kempe.

Sylvester's papers on geometry are not numerous. In 1850 there are elementary papers on the intersections and contacts of conics, and an "instantaneous demonstration of Pascal's theorem." In 1851 there is an important paper on the "contacts of lines and surfaces of the second order," in which he discusses the reduction of a pair of quadratic forms U, V in two, three or four variables, notes explicitly the invariance of the highest common factor of the minors of order r of the determinant of $U + \lambda V$, and makes use of the multiplicity of the linear factors in classifying the forms. This work forecasts in a manner the more complete theory of Weierstrass of 1868. In 1854 there is a supplementary note giving certain corrections. In volume I, pages 378–381, is given Sylvester's paper of 1852, in which he enunciates his "Law of inertia for quadratic forms," which states the invariance of the number of positive coefficients in the form $\sum A_i \zeta_i^2$ to which a real quadratic form may be reduced by a real linear transformation. His Probationary lecture on geometry, delivered in 1854, is given on the opening pages of volume II. On pages 236–244 are reproduced Sylvester's notes in the *Comptes Rendus* for 1861 on the involution of six lines in space, "double-sixes" of lines, and the construction of the 27 lines on a cubic surface. Two years later he considered questions of centers of gravity and the "principles of barycentric perspective."

Five notes or papers on the successive involutes to a circle appeared in 1868 and 1869. Adopting Cayley's name cyclode for the nth involute of a circle, Sylvester treated at length the reducible cyclodes in his paper in the *Proceedings of the London Mathematical Society* for 1869. He pursued this subject with the greatest enthusiasm. He remarks that "As crystallography was born of a chance observation by Haüy of the cleavage-planes of a single fortunately fragile specimen, and the theory of invariants owes its existence to a solitary individual accidentally encountered and put on record by Eisenstein, so out of the slender study of the Norwich spiral has sprung the vast and interminable calculus of cyclodes, which strikes such far-spreading and tenacious roots into the profoundest strata of denumeration, and by this and the multitudinous and multifarious dependent theories which cluster around it, reminds one of the scriptural comparison of the kingdom of heaven to a grain of mustard-seed," etc. He concludes his article with a formidable list of prerequisites to the mastering of his exposition of the theory.

The theory of numbers proved attractive to Sylvester throughout his sixty years of scientific activity. His first notes related to Wilson's theorem. In 1847 he discussed certain ternary cubic equations (recurring to the subject in 1879 and 1880). In the first of this series of papers, he took occasion to correct some errors in an earlier, hastily written, communication to the Institute of France. In the second of the series, he states that certain of his friends on the continent have "complimented his powers of divination at the expense of his judgment, in rather gratuitously assuming that the author of the theory of elimination was unprovided with the demonstrations, which he was too inert or too beset with worldly cares and distractions to present to the public in a sufficiently digested form. The proof of whatever has been here advanced exists not merely as a conception of the author's mind, but fairly drawn out in writing, and in a form fit for publication." The third of the series is entitled "On the *general* solution (in certain cases) of the equation $x^3 + y^3 + Az^3 = Mxyz$." Sylvester begins as follows, "I shall restrict the enunciation of the proposition I am about to advance to much narrower limits than I believe are necessary to the truth, with a view to avoid making any statement which I may hereafter have occasion to modify." In 1854 there was a further note on Wilson's theorem. In 1857 there were two papers on the partition of numbers, another on the resolvability of any integer into the sum of four squares, and a paper developing an idea of Eisenstein. The next year he wrote "On the Problem of the Virgins, and the general theory of compound partition," declaring himself at a loss to conjecture why the solution in integers of two simultaneous equations with an indefinite number of variables should be referred by Euler to "the rule of the virgins," unless it has some mystical reference to the alligation of the coefficients of the two equations; he quotes from some correspondence with De Morgan on this point. In 1859, Sylvester delivered at King's College seven lectures on the partitions of numbers. Outlines of these lectures were circulated privately at the time, but were first published in 1897 without revision by the author in the *Proceedings of the London Mathematical Society*. These outlines appear in the present volume II, pages 119–175. In the example on 120 there seems to be no reason for the omission of the partition 3, 1, 1. Cayley and Sylvester appear to have become interested in the subject on account of its application to invariants; the subject had previously engaged the attention of Euler, Waring, De Morgan, Herschel, and others. Since Sylvester's elaborate memoir on partitions in volume 5 of the *American Journal of Mathematics* belongs to a later period than the papers under review, the entire subject may appropriately be left to the care of the reviewer of the subsequent volumes. In 1860 and 1861 there were ten short papers on topics in the theory of numbers. In 1871 he discussed the Goldbach-Euler problem of the partition of an even number into two primes (recurring to the subject in later years), also some special cases of Dirichlet's theorem on the primes in an arithmetical progression.

The theory of substitution groups engaged Sylvester's attention in 1861. There were two notes in the *Comptes Rendus* relating to the number of substitutions of a certain kind. The note in the *Philosophical Magazine* related to an unsymmetrical six-valued function of six letters whose discovery had been ascribed by Cauchy to Hermite; Sylvester lays claim to priority in view of his paper of 1844 in the same journal, "On the principles of combinatorial aggregation."

Questions on tactic, a subject pursued nowadays in view of the applications in the algebraic theory of equations and in the foundations of geometry, were considered by Sylvester in the two papers last cited and three further papers of 1861 in the same journal. The results bear on substitution groups on nine letters and indirectly on the fifteen schoolgirl problem.

To the theory of determinants Sylvester made some well-known contributions which appeared in the *Philosophical Magazine* for 1851, and in the *Cambridge and Dublin Journal* for 1853. H. F. Baker, the editor of the present volumes, gives another exposition in present-day notations of Sylvester's main theorems on determinants (note to volume I).

Linear equations in finite differences were considered by Sylvester in 1862 in the *Comptes Rendus* and the *Philosophical Magazine*. In 1869 he gave in the latter journal "The story of an equation in differences of the second order."

Sylvester's dialytic method of elimination has passed into the very elements of mathematical instruction. It appeared in the *Philosophical Magazine*, 1840, pages 132, 379; 1842, page 534; 1851, page 221.

Sturm's functions were among the first objects of Sylvester's interest, and continued long to engage his attention. In 1841 he proceeded "to lay bare the internal anatomy of these remarkable forms." Divesting them of square factors, he obtains expressions in terms of the roots,

$$f_2 = \sum(k-l)^2(x-a)(x-b)\cdots(x-h), \text{ etc.},$$

which Cayley characterized as "singularly elegant." In resuming the subject in 1853, he built up a more general theory which relates, not merely to $f(x)$ and $f'(x)$, but to any pair of polynomials $f(x)$ and $\phi(x)$. To these he applies Sturm's process of the greatest common measure and obtains functions which serve to indicate the manner in which the roots of $f(x) = 0$ are intercalated among the roots of $\phi(x) = 0$. This paper on the "syzygetic relations," etc., is perhaps Sylvester's most elaborate contribution; it concludes with a seven page "Glossary of new or unusual terms, or of terms used in a new or unusual sense, in the preceding memoir." Many of these terms are familiar to present students of invariant theory. Sylvester has said: "Perhaps I may without immodesty lay claim to the appellation of the mathematical Adam, as I believe that I have given more names (passed into general circulation) to the creatures of the mathematical reason than all other mathematicians of the age combined." It has been remarked by Forsyth that Sylvester drew almost entirely upon Latin for new names, while Cayley as invariably drew upon Greek. To the same year, 1853, belong the two papers "On a remarkable modification of Sturm's theorem," *Philosophical Magazine*, where a certain series of quotients, arising from the continuedfraction development of f'/f is shown to be signaletically equivalent with Sturm's series of residues; also a paper in which these quotients are expressed in terms of the linear factors of $f(x)$. "Guided by an instinctive sense of the beautiful and fitting, in a happy moment I have succeeded in grasping this much wished for representation, with which I propose now and for ever to take my farewell of this long and deeply excogitated theorem."

Newton's rule for the discovery of imaginary roots and an inferior limit to their number had remained undemonstrated, notwithstanding the attempts by Maclaurin, Campbell, Waring, Euler, Lagrange, and others. In 1864, Sylvester published a memoir of about a hundred pages in which he established the rule for equations of degree five or less, and gave "a complete invariantive determination of the character of the roots of the general equation of the fifth degree." The next year he announced a proof for the degrees six and seven of this rule "so long the wonder and opprobrium of algebraists." In June of the same year 1865, Sylvester was rewarded by the discovery of a general proof and obtained a generalization of the rule. He contests with characteristic vigor the priority claim of J. R. Young. In 1866, Sylvester made a slight improvement in the statement of the generalized rule. The next year he presented "Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile work, and the theory of numbers." MacMahon states that one anallagmatic design by Sylvester was put down in the hall of the Junior United Service Club in Charles Street, Haymarket, but was unfortunately removed later whilst the hall was undergoing repair.

Reference should also be made to the elementary proof of Newton's rule which Sylvester gave in 1871. The theory of multiple roots had been investigated by Sylvester in 1841 and 1852; his theory for the limits to the real roots, given in 1853, connected with his investigation of Sturm's functions.

Canonical expressions for binary forms of even and odd degrees were first given by Sylvester in 1851 in three papers, of which the second was printed privately. To him the canonical form is "the very palpitating heart of the function laid bare to inspection." Cayley had previously reduced the general quartic to $x^4 + 6mx^2y^2 + y^4$. Employing the latter form, Sylvester proved in 1853 that every invariant of the general binary quartic is a rational integral function of the two well-known invariants of degrees 2 and 3, and similarly that Aronhold's S and T form a fundamental system ("scale," according to Sylvester) of invariants of the ternary cubic form. The subject of canonical forms has since been extended by Gundelfinger, Hilbert, and others.

The name invariant was introduced by Sylvester in 1851, *Philosophical Magazine*, II, page 396. Cayley's term had been hyperdeterminant.

Following the discovery of special invariants and covariants by Boole and Eisenstein, Cayley undertook a systematic study of invariants in 1845, and discovered processes for building invariants. In 1850 Hermite introduced his formes adjointes. These were obtained independently and from a simpler standpoint by Sylvester in 1851 in his brief paper, "On the general theory of associated algebraical forms"; he employs the term contravariants. In the series of papers in the *Cambridge and Dublin Mathematical Journal*, 1852–1854, entitled "Principles of the calculus of forms," Sylvester carried the theory of invariants far beyond its previous stage, introducing concepts and and processes of first importance. Special mention must be made of his general processes for the construction of contravariants and concomitants, later designated as Überschiebungen by Gordan and made by him the foundation of the theory. Sylvester introduced and developed the class of concomitants of a system of forms known as combinants. In connection with Cayley, he developed the theory of commutants. Of the very few misprints noted, we may cite that for $1 \cdot 2 \cdot \cdots \cdot n$ on pages 305 and 306 of volume I.

Sylvester's extensive papers on reciprocants belong to a period subsequent to that covered by the present two volumes.

The tentative classification of the papers under review according to the topics applied mathematics, geometry, theory of numbers, substitution groups, tactic, determinants, finite differences, elimination, Sturm's functions, Newton's rule for imaginary roots, canonical forms and the theory of invariants, was employed by the reviewer to secure continuity of thought, even at the expense of historical sequence, and with the hope of calling the attention of specialists to Sylvester's papers in their fields. For the latter reason, mention has been made of many of his minor papers. Sylvester's most important contributions related to the theory of equations and invariants. Since his work in these subjects is so well-known, it was deemed unnecessary to go into details as fully as would be warranted, but quite sufficient to enumerate landmarks which would recall Sylvester's greatest achievements.

L. E. Dickson.

HILTON'S FINITE GROUPS.

An Introduction to the Theory of Groups of Finite Order. By Harold Hilton, M.A. Oxford, Clarendon Press, 1908. xii + 236 pp.

The theory of groups, which was first developed because of its relation to the solution of algebraic equations, today enters to a greater or less extent into the structure of many other departments of mathematical thought, notably the theories of numbers, automorphic functions, algebraic substitutions, geometric transformations, and differential equations. Accordingly there is need of textbooks which will present the theory of abstract groups, not merely from the standpoint of permutation groups, but with a view to its many-sided applications to other concrete groups as well.

The book under review is evidently an attempt — and a very successful one it is — to meet this need, so far as groups of finite order are concerned. Heretofore the student has been obliged to depend, for the most part, either on books in which only abstract groups and permutation groups were considered, or on those in which groups were a secondary consideration in comparison with their applications and were therefore inadequately treated. In accordance with the well-known pedagogical principle that the concrete should always precede the abstract, it would seem that the theory of abstract groups should be copiously illustrated by examples chosen from the various kinds of concrete groups.

Moreover, for the beginner there is available plenty of illustrative material that is extremely elementary and easily comprehended. For instance, the additive and multiplicative groups of ordinary numbers, including the case where they are roots of unity; groups of translations; groups of rotations of solids like the cylinder and the regular prism; groups generated by reflections in the coordinate planes of a rectangular set; the complement and supplement group in trigonometry; the cross-ratio group of order 6; and the groups of classes of congruent integers with respect to a prime modulus, under addition and multiplication, respectively. All of these and many more like them are employed to advantage in the present text. Six of the earlier chapters are devoted entirely to concrete groups of three general kinds, permutation groups, substitution groups (chiefly groups of linear substitutions), and groups of movements. Throughout the remainder of the volume the contents of these chapters are constantly drawn upon for examples with which to illustrate and verify the general theorems.

In the development of the abstract theory a very careful and logical sequence of theorems is employed, long and intricate proofs are avoided, progress is made by short steps, and no step is taken before it is needed. As an instance in point the group concept itself is not introduced until the ground has been cleared for it in the first four chapters. At the end of each step the author takes a fresh look around at the mathematical landscape opening before him; thus very little of the scenery escapes him, so far as he travels. The only omission of interesting elementary material that occurs to me is that of the determination of the distinct abstract groups of low order, as given, for instance, by Levavasseur in his Enumérations.

The reader will be struck by the condensed style and compact notation used in the book and by the unusual arrangement of its contents, at least two-thirds of which, I should judge, is in the form of examples; and forty-three pages at the end are devoted to "hints for the solution of the examples." Some of these examples are simple illustrations of the foregoing theory or easy steps in advance, but others are more difficult and would seem to require, from the beginner's standpoint, more extensive hints for their solution than those given. Indeed, to judge by some of the examples, one might almost suppose that the book was intended to be neither a textbook for beginners nor a treatise, but rather an encyclopedic compendium of theorems without proofs. But this is by no means the impression produced by the volume as a whole. Many of the examples are referred to in the proofs of later theorems and are, therefore, necessary links in the chain of reasoning. In short, the book is an extreme example of the use of the Socratic method in the writing of textbooks, but nevertheless, in the hands of an energetic student it will certainly prove valuable and stimulating.

The reader will note that the chief properties of the elements which go to make up groups are studied in detail before their combination into groups is introduced. The term element, as defined on page 1 is, therefore, restricted to those mathematical entities which are capable of combining into groups. For instance, not all singular matrices (those whose determinants vanish) are elements in this sense.

A few elementary theorems are given on infinite groups and semigroups. In regard to the latter it does not seem to have been noticed that if A is an element of infinite order, then according to any of the definitions given by De Séguier, Dickson, or Hilton, respectively, a semigroup can be formed by selecting powers of A in very unusual and peculiar ways, for instance, by selecting the 10th, 15th, 20th, 21st, and all higher powers of A. In order to avoid such oddities it would only be necessary to make an additional postulate to the effect that if ν is the greatest common divisor of λ and μ then a semigroup which contains A^λ and A^μ must also contain A^ν.

The very latest discoveries, so far as they relate to the more fundamental parts of the subject, are embodied in the book. It is significant that commutators, which were first applied to finite groups by Miller in 1896, are introduced on page 4 and freely used throughout the book. As an evidence of the increasing vogue of the Galois field, the reader will observe that it is defined in the third chapter and that its properties are constantly used for illustrative purposes in the later chapters; also that such an elementary thing as the field of classes of congruent integers with respect to a prime modulus is defined merely as a special case of a Galois field. In examples 15–18, page 30, and example 3, page 32, certain theorems are stated concerning equations and polynomials whose coefficients are evidently intended to be integral marks of a Galois field, whereas they are incorrectly stated to be any marks of the field.

There is another oversight in example 11, page 70; it is true, as stated there, that an infinite abelian group is the direct product of a subgroup H formed by the elements of finite order and a subgroup K whose elements (besides 1) are of infinite order, but K does not, as stated there, contain *all* the elements of infinite order. The same slip was made by De Séguier.

It is to be noted that the definition of independent elements given on page 55 is not the usual one. According to this definition two elements of order 4 in the quaternion group, which are not inverse to each other, are independent, whereas they are usually regarded as mutually dependent, because they have the same square.

The condensation of the author's style sometimes tends to produce obscurity. On page 90, in examples 7 and 8, the reader is evidently supposed to understand that transitive groups are referred to, because the general subject of the section is k-ply transitive groups; but the insertion of the adjective transitive would seem to be required in order to avoid ambiguity.

Throughout the literature of the theory of groups there is a very general lack of precision in the use of the term subgroup; sometimes it includes identity and the entire group and sometimes not. Thus apparent contradictions are apt to arise. A striking instance is to be found on page 139 of Hilton's book, where example 3 contains the theorem that the central and commutant of any group are characteristic (subgroups), while the example just before it states that a simple group has no characteristic subgroups!

The group of automorphisms (or isomorphisms) of a given group is advantageously defined on pages 136–137 in a purely abstract manner and not by means of its representation as a permutation group, as is usually done. In the last chapter a very brief introduction to the theory of group characteristics is given, and the subject is approached from the proper direction, namely, that of the representation of an abstract group as an irreducible linear substitution group. This route leads directly to the heart of the matter and makes it easily accessible, as Schur has shown in his "Neue Begründung."

In nomenclature there are a number of improvements upon existing usage. Thus "prime power group" replaces "group whose order is a power of a prime"; "commutant," "central," "dicyclic," and "normaliser" are apparently borrowed from De Séguier; "permutation" replaces the commoner word "substitution," in order to avoid confusion with the more general operation denoted by the term substitution; "point group" is used to denote a group of geometric movements that leave a given point fixed, and not, as in some of the older books, an "equivalent system of points" (and therefore not a group at all) which are carried into one another by the movements of a group. Since the word "class" has been much overworked, it would seem that the author might profitably have followed his own suggestion and used "speciality" instead. In the use of the expression "normal subgroup" Weber's example is followed. The useful distinction recently made between isomorphism and automorphism (sometimes called holomorphism) is preserved; but it is not evident why an un-English coinage like "not-square" should be retained, when the well-recognized prefix "non" is so immediately available.

The following misprints were found: page 44, § 12, line 5, "the single point P' corresponds to P" for "P' corresponds to the single point P"; page 53, example 17, (ii), "some constant" for "constants"; page 64, last line, "g" for "h"; page 140, line 2, "$2n^2$" for "$2n^4$"; page 189, § 2, example 8, (i), "x" and "n" are interchanged; page 212, § 9, example 4, (ii), "$\Delta = 0$" for "$\Delta \neq 0$."

There is a good index and an appendix containing a list of problems still unsolved. The book can be heartily commended for three principal reasons, its embodiment of the results of the latest research, its carefully arranged sequence of theorems, and the emphasis which it lays on applications and illustrations.

ARTHUR RANUM.

SHORTER NOTICES.

Histoire des Mathématiques. Par W. W. ROUSE BALL. Edition française revue et augmentée. Traduite sur la troisième édition anglaise par L. FREUND. Tome deuxième. Avec des additions de R. de Montessus. Paris, A. Hermann, 1907. vi + 271 pp. 8 francs.

IT was natural to hope that Lieutenant Freund would succeed better with his second volume of Ball than he did with the first,* since he received warnings enough at the hands of numerous reviewers. In a way the hope has been realized, although in the mere matter of translation and of accuracy there is, unfortunately, no improvement.

The valuable part of this second volume, as it appears in its French form, lies in the additions made by M. R. de Montessus, 'docteur ès-sciences mathématiques, lauréat de l'Institut.' These are numerous, particularly with respect to French mathematics, and they materially increase the helpfulness of the work as a book of reference. The period treated, being that beginning with Newton and running to the close of the nineteenth century, is one of great interest from the standpoint of the Paris mathematicians, and it was natural that a Cambridge writer could hardly do full justice to the labors of writers like l'Hospital, Varignon, De Montmort, D'Alembert, Laplace, and several others whose works Mr. Ball has briefly described. Other continental authors that are hardly mentioned at all in the English edition have fairly adequate biographical notices in the translation, and on this account the additions of Dr. de Montessus are particularly helpful. Among the added names are Meusnier, Lhuillier, Agnesi, Lacroix, Malfatti, Delambre, Montucla, Laurent, Cournot, Genocchi, Betti, Puiseux, Bouquet, Codazzi, Faà di Bruno, Catalan, Brioschi, Casorati, Halphen, Wronski, Bertrand, Laguerre, Stieltjes, Clebsch, Dupin, Chasles, Bellavitis, Cremona, Beltrami, Poinsot, Tisserand, and several others, none of whom are treated, or who at most are merely mentioned, in the English edition. Curiously enough it required a French translation tó mention Waring, who one time held the Lucasian professorship, and whose "identity" would seem to have merited the insertion of his name, at least, in such an English work.

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* See review in the BULLETIN, vol. 12, p. 309.

Unfortunately for the best use of the book, there is no index, and the table of contents is not sufficient to make up for the omission. Of all works, one of this nature, to which a student must often refer for a single topic or name, should have a complete index covering both volumes.

It has been said that the translation is not much of an improvement upon that of the first volume. If any proof of carelessness is needed, a glance at the table of 117 errata discovered by the author himself should suffice. But he has by no means covered the list, for even the casual reader will find scores of others. For example, for logarihmotechnica (page 14) read logarithmotechnia, and on the same page for x^{m-n-1} read x^{mn-1} twice; for $1-y'^2$ read $1-y^2$ (page 28); in Américan, an English word, drop the accent (page 53); for ${V^2}_2$ read V_2^2 (page 54); for $\sin x + i \cos x$ read $\cos x + i \sin x$ twice on page 72, thus correcting an error that still exists in the fourth English edition, which has appeared since the translation, and on the same page change the date of De Moivre's death to 1754; spell the French for Edinburgh uniformly, both Edimbourg and Edinbourg appearing on page 73, and elsewhere; change Simpson's birth year from 1610 to 1710 (page 77); for "Arithmetic of lines" read "Arithmetic of sines" (page 82); for Englisch read English (page 89); for 1771 read 1781 (page 122). These are only typical of a considerable number of similar errors that will strike the reader, and they go to show how carelessly the work has been performed. We should be thankful for the added matter, but we might reasonably have hoped for a translation with a minimum instead of a maximum number of errors.

DAVID EUGENE SMITH.

Geschichte der Mathematik. I. Theil. Von den ältesten Zeiten bis Cartesius. Von Dr. SIEGMUND GÜNTHER. Leipzig, G. J. Göschen'sche Verlagshandlung, 1908. 8vo. 56 figures. viii + 427 pp. 9 Marks.

THE great interest manifested of late in the history of mathematics, evidenced by the fourth volume of Cantor, the enlarged form of the *Bibliotheca Mathematica*, the success of the *Abhandlungen*, the Encyklopädie, and the translations of Ball, assures a hearty welcome to a work like this of Professor Günther's. If the undertaking had been merely a rewriting of old material or a condensation of Cantor, it would not be without its value, but since it is the product of a man with the historical ability and the reputation of Dr. Günther it is certain to take rank as an authority in its field.

The work was undertaken originally in collaboration with Professor von Braunmühl, the plan being that the author of the present volume should carry the work to the middle of the seventeenth century, and von Braunmühl from there on. What the effect of the recent death of the latter may have on the success of the undertaking it is unpleasant to consider, but it is to be hoped that the manuscript of the second volume was completed before the world was deprived of the services of one whose history of trigonometry will remain a standard authority for a long time to come.

This first volume consists of twenty chapters, as follows: Number and measure as primitive concepts of mankind, Mathematics in Mesopotamia, Mathematics in Egypt, Mathematics in China and ancient India, The prealexandrian period in Greece Classical antiquity, Greek mathematics between Apollonius and Ptolemy, Roman mathematics, The decline of Greek mathematics, Byzantine mathematics, Mediaeval Hindu mathematics, The early Arab period, The later Arab period, The position of science in the Church and Court schools of the Christian middle ages, Leonardo of Pisa, Mathematical teaching and progress in the later middle ages, The reform period of Peurbach and Regiomontanus and their followers to 1500, General characteristics of the 16th and early 17th centuries, The arithmetical sciences between 1500 and 1637, and The geometric and mechanical sciences in the same period. The work is followed by an index of names, the general index being presumably, and meantime unfortunately, left until the close of the second volume. There is also a helpful bibliography of seven pages, far from complete but valuable to beginners.

The first thought that may occur to a reader will relate to the reason for writing such a work, in view of the rather extensive literature that we have already upon the subject. The question is answered, however, as soon as he begins to examine the book. For one thing, mathematical history has advanced since Cantor wrote his first two volumes, and new material is now available. This has been used sufficiently to be in itself a warrant for such a treatise. For example, Kugler's work of 1900 on the lunar tables of Babylon, Hilprecht's work on the Nippur excavations (1904, but not that of 1906), von Braunmühl's contributions, Tannery's latest articles — these and others of similar nature have contributed to make the book more up to date than any similar work available. Furthermore, greater seriousness of purpose and maturity of judgment characterize this history than are found in any other of the smaller treatises; that is to say, there is less of the anecdotal and more of the historical and mathematical than would appear in a work like Ball's, for example. On the other hand, the book has not the heaviness of Cantor, but preserves the readable style that is found in the author's other treatises.

In spite, however, of the fact that Professor Günther has given us the best elementary hand book of early mathematical history that has yet appeared, it is not without its points of weakness, partly from lack of sources, partly from errors of judgment. It seems unfortunate, for example, that one should attempt to write upon the history of Chinese mathematics without knowing more than the little given by Biernatzki, and by Biot in his translation of the Le Tcheōu Ly. To be sure the author mentions Wylie, but he has evidently never read his best contributions; while as to the works of Williams, Vissière, Knott, and Hyashi (and of course Endo's work in Japanese) he is apparently wholly ignorant. In the same line of bibliography one can readily understand that the author would be justified in omitting the older authorities if he thought best, but why he should mention Kaestner, for example, and omit Montucla, Libri, and Bossut, is not so easily explained. So in general, while the bibliography is helpful, it does not seem to be as complete as might reasonably be hoped, nor as well selected as it should be for its present extent. Had Vissière's excellent monograph been read, for instance, the too sweeping assertion about the antiquity of the swan pan (page 36) would not have been written, and had any serious study been made of the voluminous literature relating to the Yih-king more doubt would have been cast upon the Leibniz hypothesis with respect to the connection of the trigrams with a binary scale. The paucity of sources consulted is also manifest in the brief mention of Tschu schi kih (Chu Shi-ki) and his knowledge of the binomial coefficients. Now it is true that the work of this author appeared "bald nach 1300," more exactly in 1303, and that it was quite epoch-making, but it is best known for the use made of several monads (unknowns) in systems of equations, and for the ingenious symbolism for polynomials. He was only one of three great writers, nearly contemporary, who created a renaissance of mathematics in China about that time, the other two being Tsin Kew-chaou, who introduced the monad as the symbol for the unknown and who did very creditable work on higher numerical equations, and Li-yay Jin-King, neither of whom are mentioned.

What is here said concerning Chinese mathematics is merely typical. It is easy to see that Professor Günther has used the best of the common secondary sources, but that rarely has he gone to those that may be called primary. Thus on the Greek abacus, he assumes that the tradition of its common use is to be accepted, when as a matter of fact it is extremely doubtful, and with respect to the similar question in Rome he asserts that we have no examples of such an instrument extant, while we have several that may quite possibly go back to classical times. As to the early printed arithmetics he cannot have consulted many original sources. For example, Licht's work was published without date, but probably about 1500, and not in 1514 as he asserts; Stromer's work appeared in 1504, and in two editions before the first one cited; the date of the Protomathesis of Finaeus is more properly 1530–32 than 1532; Ringelberg's Opera (not Chaos mathematicum) appeared at Leyden (not Lyons) in 1531; Ciruelo's Cursus quatuor appeared at Paris as well as at Alcalá in 1516; Ramus's 1569 work was really entitled Arithmeticæ libri dvo: Geometriæ septem et viginti, and reference should be made to his earlier work of 1555; the Opuscula mathematica of Maurolycus was published at Venice instead of Palermo, in 1575; the first edition of Moya was 1562 instead of 1609, which makes quite a difference in the argument on page 343; Nuñez's work appeared in 1564, the 1567 edition being the second. This list is closed simply because it would be unprofitable beyond this point, and not because it could not be greatly extended. It shows that Professor Günther has been unfortunate in his secondary sources or careless in his use of primary ones.

Independently of such errors, however, the fact remains that the book is the best that we have of its particular type.

DAVID EUGENE SMITH.

Manuscrits de Evariste Galois. Publiés par JULES TANNERY. Paris, Gauthier-Villars, 1908. 67 pp.

THE articles published by Galois related to periodic continued fractions, the algebraic solution of equations (summary of results), and the introduction of quantities (now known as Galois imaginaries) determined as the roots of irreducible congruences. These brief articles appeared during the years 1828–1830 in the *Annales de Mathématiques* de M. Gergonne, and the *Bulletin des Sciences mathématiques* de M. Férussac. They have been reprinted as pages 1–23 of Œuvres mathématiques d'Evariste Galois, Paris, 1897. Pages 25–61 of the latter book give a reproduction of certain posthumous papers by Galois, which first appeared in Liouville's *Journal de Mathématiques*, volume 11 (1846), pages 408–444, fourteen years after the tragic death of the author. These posthumous papers are the celebrated letter to Auguste Chevalier, his fundamental memoir on the conditions for the solution of algebraic equations by radicals, and an incomplete manuscript on solvable primitive equations.

Chevalier prepared for publication the manuscripts entrusted to him by Galois and placed them in the hands of Liouville. The latter published the three papers just mentioned, promising in a footnote to add later some other fragments extracted from the papers of Galois, which, although without great importance, could be read with interest by geometers. However, this additional manuscript by Galois is now published, for the first time, by Tannery, who gives in a most faithful manner a minute account of all the Galois manuscripts, points out as far as possible the few annotations by Chevalier, and indicates by detailed references the occasional divergence between the original manuscript by Galois, the copy by Chevalier, the text by Liouville, and that of the Œuvres. On page 15, line 7, the correct reading 1, 2, 3 should replace 1, 3, 3, and not 1, 1, 3.

In contrast to the statement in Picard's Introduction to the Œuvres, "les deux Mémoires qu'il présenta à l'Académie des Sciences ayant été perdus," Tannery has now made it clear, on pages 5 and 6, that the manuscript presented to the Academy by Galois was not lost, but is precisely the one which passed into the hands of Chevalier, thence to Liouville, and has finally been given to the Academy of Sciences by Liouville's daughter. What was rejected in 1831 has now been received with honor! The manuscript bears the names Lacroix and Poisson of the members of the commission which passed upon this work of Galois on the solution of equations by radicals, now universally admitted to be one of the greatest triumphs of the human intellect. The verdict of the commission, however, delivered by Poisson in large handwriting, was simply "vu." The manuscript shows Galois's comment: "Oh! chérubins." There also appears in Liouville's writing "Rapport du 4 juillet 1831"; but he refrained from publishing the work until 1846 and then with all references to its earlier presentation to the Academy carefully deleted, though not on the first set of proofs still preserved. Liouville was the most conscientious of editors and there were details not clear to him in the manuscript of Galois; in fact, Liouville made various additions, afterwards suppressed, in an attempt to bring the manuscript into conformity with his precept "of being transcendentally clear." All honor to Liouville for recognizing the fundamental character of Galois's work and for preserving it to the profit of all ages.

The text of the manuscripts of Galois not previously published occupy over forty pages, including some work while still in school, and several introductions to memoirs merely planned, or else begun but left unfinished. The majority of the manuscripts are fragmentary notes bearing on the solution of equations by radicals. The second sentence on page 22 will be of interest. "Si maintenant vous me donnez une équation que vous aurez choisie à votre gré et que vous desiriez connaître si elle est ou non soluble par radicaux, je n'aurai rien à y faire que de vous indiquer le moyen de répondre à votre question, sans vouloir charger ni moi ni personne de le faire. En un mot les calculs sont impracticables." In defense of his method, however, he indicates that in practice one knows in advance properties of a proposed equation which will aid in applying his method. The same point may be raised in the application of Lie's theory. One does not make headway with equations, whether algebraic or differential, if selected at random or stripped of the information accompanying their origin.

M. Tannery has executed admirably the task he laid before himself. He has given to the world an impartial historical work of great interest, a necessary companion to the Galois text already available.

L. E. Dickson.

Diophantische Approximationen. Eine Einführung in die Zahlentheorie. Von HERMANN MINKOWSKI. Leipzig, Teubner, 1907. viii + 235 pp.

THE title of this original and highly interesting work is applied by the author to extensive classes of inequalities to be satisfied by integral values of the unknowns, the constants being any quantities; just as hitherto equations of the corresponding type have been designated with the name of Diophantus. In the first chapter, the author considers the rational approximation to one or more arbitrary real numbers, gives a new proof of the existence of integral solutions of $sx - ry = 1$, s and r being relatively prime integers, and establishes the theorem that there exist integral values, not all zero, of three variables such that each of the three given linear forms of determinant unity has a value not exceeding unity numerically. In the proof, due to a suggestion by Hilbert, the author introduces the concept of the minimum of a given system of linear forms for integral values of the variables and shows that the minimum changes continuously under continuous variation of the coefficients of the forms. An earlier proof by Minkowski, based upon geometric considerations, was given in his Geometrie der Zahlen, Leipzig, 1896.

Chapters 2 and 3, forming about half the book, relate to lattices; a Zahlengitter in two or three dimensions is defined to be the totality of sets of integral values of two or three variables, with the obvious geometric representation by a lattice. Here the author gives a detailed account of his theory of convex figures or solids having a center, in their relations to the lattice. In particular he discusses the following question: How may an infinitude of given congruent central convex solids be placed in parallel positions, such that no two of the solids overlap, such that the centers form a three dimensional lattice, and such that the space not filled by the solids shall be as small as possible? The problem is discussed at length and definitive analytic criteria are deduced (page 104). The text carries the discussion to a final conclusion for spheres and for ellipsoids and applies the results to the immediate derivation of a complete theory of positive ternary quadratic forms, obtaining in particular the results of Gauss and Seeber.

The second half of the text develops the more fundamental parts of the classic theory of algebraic numbers from the author's novel standpoint. The example of cubic number fields $K(\theta)$ is usually chosen to secure simplicity in the geometric interpretations. The integral algebraic numbers of $K(\theta)$ correspond uniquely to the points of a lattice. The existence of a basis for the integral algebraic numbers follows at once geometrically. An inferior limit for the discriminant of a number field is established; except for the field of rational numbers the discriminant exceeds unity. Within the lattice which pictures the totality of integral algebraic numbers there exists (page 156) a lattice whose points picture uniquely the numbers of any given ideal. There follows geometrically the existence of a basis of the ideal. The introduction of lattices not only secures a highly serviceable geometric setting for the algebraic theory, but in connection with related concepts from the author's Geometry of Numbers enables him to establish some fundamental theorems on algebraic numbers which have not been demonstrated without this geometric theory. The final chapter on the approximation of complex quantities by means of numbers of the fields defined by a cube root or a fourth root of unity is the work of Dr. Axer, who also otherwise aided in the publication of the book.

Since the author has given an elementary, entertaining account, both in geometric and arithmetic language, of some important original results as well as the salient features of a classic theory, but presented in a novel manner, his work is deserving of the attention of the very widest circle of readers.

L. E. DICKSON.

Die Lehre von den geometrischen Verwandtschaften. Zweiter Band; die eindeutigen linearen Verwandtschaften zwischen Gebilden zweiter Stufe. By RUDOLF STURM. Leipzig and Berlin, B. G. Teubner, 1908. vi + 346 pp.

THE present volume of Professor Sturm's treatise has about the same programme for two dimensions as the preceding one has for one,* except that non-linear transformations are only considered incidentally. This part is not provided with an index nor a preface, but contains a glossary of 131 technical words besides those found in the preceding one, and the table of contents of all four volumes.

The first chapter (pages 1–218) is concerned with collineation and correlation in the plane, being somewhat similar in scope and treatment to parts of Reye's treatise, volumes two and three. The proof that the vertices of any two quadrangles are sufficient to define projectivity in the plane is identical with that given by Cremona (Elements of projective geometry, Art. 94), but the author acknowledges assistance from a letter from Reye. After the general properties have been considered, metrical properties, particularly vanishing lines and foci are discussed; particular cases, such as affinity, similarity, and congruence are treated at length. In the discussion of two superposed collinear fields, the existence of one real self-corresponding point is assumed. The rest of the discussion is then clear enough, but it seems odd that no proof of this fundamental theorem is given. Homology is given an important place and Hermitian collineations are treated at length. Thus far correlation has hardly been mentioned; it is introduced by duality, showing that a pair of elements may be simply or doubly conjugate. If all elements are doubly conjugate, we have a polar field. The discussion of this part is very detailed; in connection with imaginary elements it is shown that loci exist, all of whose points are imaginary, but arranged in conjugate pairs (Klein's null-partite curves). A large number of constructions of polar fields is given, including the determination of foci, center, conjugate diameters of the invariant conic, and self-conjugate triangle of a pencil or a series of conics. These ideas are now developed in the geometry of the bundle, the procedure being quite similar to that given by Reye, volume 1, chapter 15, but followed by a large number of trigonometric formulas involving relations between conjugate diameters, cyclic planes and principal axes.

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* See the review in the December number of the present volume of the BULLETIN, p. 135.

Now follows a well-arranged treatment of cyclic collineations. A number of statements are not literally true; *e. g.*, that a cyclic homology of period greater than 2 can not exist (page 162). We gradually see that the author is considering only real elements, but in other articles imaginaries are brought in without warning. The development reminds one of that of Ameseder, but his name is not included among the authors cited. Since the guiding idea is the discussion of forms having two degrees of freedom, quadric and cubic surfaces, space cubic curves and Hirst's complex are legitimate topics of discussion. The polar theory of quadrics is presupposed and a large number of theorems are given without proof, with frequent footnotes giving references to the original memoirs.

Thus far the development has been systematic and comprehensive, and a diligent student could master much of the subject without a teacher, although proofs are much more condensed and the amount of presupposed knowledge much higher than in the preceding volume. The second chapter however (pages 219–346) is concerned almost entirely with problems of enumeration, and is to be regarded as a hand-book, rather than a text. It begins with a short discussion of singular correlations, and uses the results persistently throughout the chapter.

The procedure is as follows: Given the point plane $\Sigma \equiv (A, B, \cdots)$ and the line plane $\Sigma' \equiv (a', b', \cdots)$ correlated in such a way that A has a' for image, B has b', etc., and that B, C, D are collinear, but a', c', d' form a triangle.

The pencil $A(B, C, D)$ and the range $a'(b', c', d')$ are projective, so that corresponding to any line AX will be a definite point X' on a', but other pencils, *e. g.*, $B(A, C, D)$ go into the single point $a'b'$. We can now construct the image of any point or line in either plane. The lines a' and $a \equiv (B, C, D)$ are both singular. This correlation is called axial. If the relation between Σ, Σ' be such that a', c', d' are concurrent, but B, C, D not collinear (dual of the preceding case) the central correlation with the singular points A, A' results.

There are ∞^8 correlations in the plane. If a given point P is associated with a given line p', this counts for two conditions, since two parameters are necessary to fix P. If, however, P is to be on some given line p, without more definitely fixing it, the fact that p, p' are conjugate counts for one condition. Now consider α pairs (p', P), β pairs (P', p), γ pairs (A, A'), δ pairs (a, a') given, wherein

$$2\alpha + 2\beta + \gamma + \delta = 8.$$

The number of polar fields fixed by these conditions is indicated by $(\alpha\beta\gamma\delta)_8 = (\beta\alpha\gamma\delta)_8 = (\alpha\beta\delta\gamma)_8 = (\beta\alpha\delta\gamma)_8$, since α, β and γ, δ can be interchanged by duality.

If now one condition be suppressed, in $(\alpha\beta\gamma\delta)_7$ there are ∞^1 correlations, including a finite number π central collineations, λ axial correlations, μ non-singular ones in which a given pair of points are conjugate, and ν in which two given lines are conjugate. To obtain relations among these numbers, consider two lines a, b given in Σ, and a point C' in Σ'. A line x'_a through C' and a are therefore conjugate in ν correlations. In each of these construct the pole of b, and connect it with C' by means of x'; these lines are respectively conjugate to b. Thus a (ν, ν) correspondence between the lines of C' is established, having 2ν coincidences made up of π central collineations and μ having a, b and c' conjugate points, hence $2\nu = \mu + \pi$; and by duality $2\mu = \nu + \lambda$. If the number of correlations μ in the pencil having a given pair of conjugate points be denoted by $\mu(\alpha, \beta, \gamma, \delta)_7$, we may therefore write

$$\mu(\alpha, \beta, \gamma, \delta)_7 = (\alpha, \beta, \gamma + 1, \delta)_8.$$

Finally, if two conditions be suppressed, in $(\alpha, \beta\, \gamma, \delta)_8$ are ∞^1 axial and ∞^1 central collineations. The conjugate points will describe curves, the conjugate lines will envelop others. By use of the chapter on multiple correspondence developed in the first volume it is now possible to determine $(\alpha, \beta, \gamma, \delta)_8$ in each case.

Many of the details are omitted; in order to follow the argument in the later cases of the bundle frequent use must be made of Schubert's Enumerative geometry, Hirst's article in the *Mathematische Annalen*, volume 8, and in particular to the comprehensive memoir of the author in volume 12 of the *Mathematische Annalen*. Contrary to the promise in the preface of the first volume, the treatment is synthetic throughout the volume, thus making the development quite one-sided and restricted.

VIRGIL SNYDER.

Arithmétique Graphique. Les Espaces Arithmétiques, leurs Transformations. Par G. ARNOUX. Paris, Gauthier-Villars, 1908. xii + 84 pp.

THIS book treats of arithmetic spaces which have been defined and studied by the author in two previous books written under the same general title, "Arithmétique Graphique."* The second of these was reviewed by the writer in May, 1907.† The reader is referred to this review for the definition of arithmetic spaces and for a short account of Arnoux's use of them to furnish a graphical representation in the theory of numbers.

In this third book, the author is more interested in the properties of the arithmetic spaces themselves and not so much in their applications. He has in mind the creation of what he calls a géométrie analytique arithmétique which shall be to the theory of numbers what ordinary analytic geometry is to algebra and analysis. He does not claim to have made more than a beginning in the working out of his idea, but clearly states that he has tried only to point out the possibility and the usefulness of such an attainment.

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* Les espaces arithmétiques hypermagiques, Paris, 1894. Introduction a l'étude des fonctions arithmétiques, Paris, 1906.

† BULLETIN, vol. 13, pp. 402–403.

In the book under review the author uses coordinates more extensively than in his earlier work. He finds that the general idea of geometric transformation may be conveniently applied to arithmetic spaces, and that the general linear transformation in a two-dimensional arithmetic space changes arithmetic lines into arithmetic lines; that is, it is a collineation. This is true whether the arithmetic space be infinite or modular. In an infinite arithmetic space, the domain for the coordinates of points is the totality of integers, but in an arithmetic space of modulus m the domain consists of integers, modulo m. If m is a prime number, these integers form a field or finite algebra, and consequently the analytic geometry of the space is more like ordinary analytic geometry. In particular, the transformation called inversion is possible. Arnoux calls attention to this fact.

The reviewer takes this opportunity to mention the fact that Arnoux's k-dimensional arithmetic space of prime modulus p is what Veblen and Bussey have called a finite euclidean geometry and have denoted by the symbol $EG[k, p]$.[*] Arnoux has not given the theorem that the linear transformation in a k-dimensional modular arithmetic space ($k > 2$) of prime modulus is a collineation, nor has he made any study of groups of transformations in these finite euclidean geometries.

The book closes with a chapter of applications to number theory, magic squares, and Euler's problem of the 36 officers.

W. H. BUSSEY.

Fragen der Elementargeometrie, gesammelt und zusammengestellt von FEDERIGO ENRIQUES. Deutsche Ausgabe von DR. HERMANN FLEISCHER. II Teil. pp. xii + 348. Leipzig, B. G. Teubner, 1907.

ONE of the few unique books of the last few years on elementary mathematics was the collection of related monographs on "Questioni riguardanti la geometria elementare" which appeared in 1900 under the editorship of F. Enriques. The volume under review is more than a translation, it is a thorough revision of the portion of the original that deals with the theory of geometrical construction. The other volume will contain a discussion of postulates, definitions, theory of area, the parallel axiom, and related topics.

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[*] "Finite projective geometries," *Transactions Amer. Math. Society*, vol. 7 (1906), pp. 241–259. In particular, see §7.

In every respect the German edition is an improvement on the original. The text is less involved, and assumes little more than should be acquired in an ordinary college course in the calculus. Even this degree of proficiency is unnecessary for most of the book.

The first chapter contains a discussion of the various methods of attacking original problems of construction, and follows rather closely the excellent little book of Petersen on this subject. The next ninety pages are devoted to constructions that can be performed by the use of particular instruments, as, for example, the compasses alone, where the spirit of Mascheroni's work is very clearly presented; the ruler alone, showing the nature of the projective properties of figures; the parallel ruler; and various instruments in connection with a fixed curve. It is almost a pity that Lill's very elegant constructions of cubic and higher irrationalities by means of the movable right angle are not mentioned.

The analytic side of the subject is not neglected, and a clear demonstration of the impossibility of making constructions of higher order than the second with ruler and compasses is presented. The construction of the 17-gon is, perhaps, given more prominence than is necessary, but this is offset by the concise discussion of the various exact and approximate solutions of the classical problems of trisecting the angle, and duplicating the cube. Several approximate constructions are given for π, and the transcendence of π is demonstrated.

The book is a most welcome contribution to the literature that should be of interest to teachers in secondary schools. It makes no pretense of showing how to teach, but gives a background of fact and theory that no teacher should lack. The writer uses it as required reading in an undergraduate course for prospective teachers, and believes that most of the book can be appreciated by undergraduates. Though the style and aim of most of the chapters is for reading by professional teachers rather than professional mathematicians, at the same time the number of important ideas to which the reader is introduced is surprisingly large, and what is of as great importance, a breadth of outlook pervades the volume that teachers will certainly appreciate.

H. E. HAWKES.

Leçons de Mécanique céleste professeés à la Sorbonne. Par H. POINCARÉ. Paris, Gauthier-Villars. Tome I, 1905, 365 pp.; tome II, 1re partie, 1907, 165 pp.

TISSERAND has written an excellent four volume treatise on celestial mechanics, containing most of the classic contributions to the subject in the spirit and often in the notations of their original authors. Poincaré's Les Méthodes nouvelles de la Mécanique céleste was devoted to establishing, with the rigor of modern mathematical methods, the existence of various kinds of periodic orbits, to determining their properties, to proving the non-existence of new uniform integrals analytic in the masses and the existence of asymptotic solutions, etc. These profound and important researches have almost nothing in common with earlier investigations either in method or subject matter. The Leçons of Poincaré occupy ground between that covered in Tisserand's treatise and in Les Méthodes nouvelles. They are largely expositions of certain parts of standard theories, but the methods of development are those which the author considers as best suited to reaching the desired end regardless of what other writers may have used. It is doubtful if Poincaré could have compelled himself to take the little steps his predecessors have sometimes taken, or to follow their often circuitous paths. At any rate it is fortunate for us that we are able to see how the most penetrating and brilliant mathematician who has written on celestial mechanics reacts on some of our standard theories.

The first volume of the Leçons is devoted to the general theory of planetary perturbations. The first chapter contains an account of canonical equations, their properties, and their transformations. The canonical equations are used throughout the work. The problem of elliptic motion and the equations for the variations of the elliptic elements are developed in these variables, and Lagrange's method of variation of parameters is explained and applied. It is easily shown that the general term in the expression for either a coordinate or an element is of the form

$$\mu^a A M t^m \cos(\nu t + h),$$

where μ is a small quantity of the order of the masses, A a function of the ratios of the major semiaxes of the orbits, M a power series in the eccentricities and inclinations, and ν a linear homogeneous function of the mean motions. If $m = 0$, $\nu \neq 0$ the term is *periodic*, if $m \neq 0$, $\nu = 0$ the term is purely *secular*, and if $m \neq 0$, $\nu \neq 0$ the term is *mixed secular*. Poincaré calls α the *order* of the term, the degree of the term of lowest degree in the small quantities in M the *degree* of the term, $\alpha - m$ the *rank* of the term, and, if m' represents the exponent of the small divisor arising through integration, $\alpha - m/2 - m'/2$ the *class* of the term. For practical work the importance of a term depends upon its order, degree, rank, and class, and the classical results relating to the stability of the system are attached to relations among these properties. In Chapter V Poincaré has proved in a very simple way, considering the complexity of the subject, that (1) there are no terms of negative rank; (2) there is no mixed secular term of rank zero; and (3) in the expressions for the major axes there is no term of rank zero. These theorems have been known since the time of Poisson for the first two approximations ($\alpha = 1, 2$), and they are recalled here because Poincaré shows they are general.

Only a few of the many results and methods of interest can even be mentioned here, but the conclusions reached in Chapters VIII and IX cannot be passed in silence. These chapters are devoted to a discussion of the terms of rank zero. It is clear that while these terms may be of little importance for small values of t, they are important in determining the configuration of the system for large values of t, certainly if the series in which they occur are convergent. The original differential equations may be written in the form

$$\frac{dx_i}{dt} = \sum_{j=0}^{\infty} B_j \cos(\nu_j t + h_j).$$

Certain of the ν_j are zero. Let the values of B_j for which this is true be $B_j^{(0)}$, and those for which $\nu_j \neq 0$ be $B_j^{(1)}$. Then we have

$$\frac{dx_i}{dt} = \sum B_j^{(0)} \cos h_j + \sum B_j^{(1)} \cos(\nu_j t + h_j).$$

The first terms in the right members give rise to terms of rank zero and order one. Let the terms in $B_j^{(0)}$ of degree one be represented by $B_\nu^{(00)}$. Lagrange treated the question of stability by taking out these terms and writing

$$\frac{dx_i^{(00)}}{dt} = \sum B_\nu^{(00)} \cos h_\nu.$$

Since there is no term in the major axes of rank zero, they are constant so far as these equations are concerned. Making use of this fact, the remaining equations become linear and homogeneous with constant coefficients. Lagrange showed that all the roots of the characteristic equation of this system are purely imaginary, and inferred from this result that the solar system is stable. However, if the process of Lagrange be extended to the terms in $B_\nu^{(0)}$ of higher degree it is found that secular terms appear in the expressions of the third order.

The result obtained by Poincaré, and given also in Les Méthodes nouvelles, volume II, chapter X, is that the equations

$$\frac{dx_i^{(0)}}{dt} = \sum B_\nu^{(0)} \cos h_\nu$$

can be transformed and integrated *formally* so that the $x_i^{(0)}$ involve only trigonometric terms.

In chapter X he shows that the general problem of three bodies may be *formally* integrated in purely trigonometric series, a result established long ago by Newcomb and Tisserand.

Volume II, Part 1, of the Leçons is entirely devoted to the development of the perturbative function. The whole problem is treated with the skill of a master. From the point of view of novelty and mathematical interest chapter XX is the most important. It is devoted to a discussion of the convergence of the series for the coefficients of the development of the perturbative function. The problem is the development of a certain function $F(u, u')$, which is periodic in both u and u' considered separately, in a series of the form

$$F = \sum B_{mm'} e^{i(mu+m'u')}.$$

The $B_{mm'}$ are functions of various of the elements and are expanded as power series in certain of these parameters. They are defined by

$$B_{mm'} = \frac{1}{4\pi^2} \int_{|u|=1} \int_{|u'|=1} F e^{-i(mu+m'u')}\, du du'.$$

The singularities of B_{mn}, as functions of the parameters (let us call them $\beta_1, \cdots, \beta_n$) are found by varying $\beta_1, \cdots, \beta_n$ in F, and finding under what conditions the contours of integration can not be deformed so as to escape passing through a singularity. By a very beautiful discussion the general problem is set up, and the details are worked out in some of the simpler cases.

In conclusion, we may inquire to what class of readers these volumes will most appeal. They certainly can not be digested by American students who are just starting work in the lines to which they are devoted. While Poincaré begins at first principles and works out the results in rather complete detail, the giant strides of the master cross too wide a field of mathematics to be within the grasp of beginners in graduate work. The mathematics would all be readily intelligible to the mature mathematician, but those who were not already somewhat familiar with the subject matter would very often find themselves at a loss to know why the numerous particular artifices were employed. For one having an interest in, and knowledge of, celestial mechanics, this work will be a great source of information and inspiration. In these respects it is surpassed only by the incomparable Méthodes nouvelles, which will, I believe, indirectly revolutionize celestial mechanics even in all its practical details.

F. R. MOULTON.

Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Ärzte. Gesamtbericht, herausgegeben von A. GUTZMER. Leipzig, 1908, large 8°, pp. xii + 322.

IN a stately volume there is laid before us the complete report of the Commission appointed in 1904 by the Society of German natural scientists and physicians to examine and report upon various proposed reforms in the teaching of mathematics and and the natural sciences in Germany. The present report contains reprints of the proceedings in Cassel (1903), and in Breslau (1904), which led to the formation of the Commission, and of the proposals submitted by the Commission in Meran (1905), Stuttgart (1906), and Dresden (1907).

The central questions relative to the teaching of mathematics were taken up in the proposals of 1905, which have already been discussed in the BULLETIN * and the later reports deal with matters of smaller general interest or importance. The proposals of 1906 have been reviewed in the BULLETIN;* those of 1907, so far as mathematics is concerned, deal mainly with the scientific preparation of candidates for appointment as teachers. An interesting explanatory address by Professor F. Klein introduces this part of the report, which among many things of a local character contains others of a far wider range of interest. Its spirit cannot be laid before American readers more concisely and effectively than by citing a few characteristic recommendations of wider applicability, whose basic principles, translated if need be into our terminology, deserve as thoughtful consideration in America as in Germany.

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* Vol. 12 (1906), pp. 347–352.

It will be recalled that the preparation of the prospective teacher of mathematics in Prussia consists of two parts, the scientific preparation, attained in a minimum of three years of university study (beginning with analytic geometry and the calculus), and the pedagogic preparation acquired by two years of training in a secondary school. Only the scientific preparation is considered in the report, which points out (pages 270, 273) that when once the function concept has been made the center of the mathematics of the schools, as recommended by the Commission in 1905, the introductory university work will be a natural outgrowth and continuation of the secondary work and not separated from it by a marked gap as has hitherto too generally been the case.

It is recommended that the university student should from first to last not only hear lectures but also be constantly employed in active work himself, ranging from the treatment of small problems to independent investigations, reporting thereon before his fellows in seminars. The professors should have assistants in conducting this work (in the schools of technology in Prussia it is already customary to appoint one assistant per 30 students) and suitable libraries and working-rooms are also indispensable.

"The commission also recommends emphatically that at the end of the general studies in pure mathematics a course be given organizing the entire mathematical material according to its essential interrelations and as far as possible presenting the import of the higher branches for the different stages of school mathematics. For, in fact, experience teaches that without such a course of study, the majority of the students do not discover the inner bond that connects the various parts of mathematical science, and thus the prospective teacher loses what should be for him the real gist of his mathematical studies. To avoid misunderstanding, we add expressly that this course presupposes matured hearers, and should not be brought down to the level of those preparing to teach mathematics as a minor subject only."

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* Vol. 13 (1907), pp. 304–305.

The commission also proposes the following tentative scheme of general professional studies for prospective teachers of mathematics and physics. A common program is proposed for the minimum period of three years, specialization in one direction or the other being contemplated in case the student remains at the university longer than three years, as many do. The courses are presumably of four hours each weekly, supplemented throughout by exercises and seminars as recommended above. Little stress is laid upon the order of the courses, which must be varied to correspond with the courses offered by universities that are not able to offer all the courses of the outline each year.

Half Year.	Professional Studies			General Studies.
First.	Calculus I.	Plane Analytics.	Experimental Physics I.	
Second.	Calculus II.	Descriptive Geometry (with projective geometry).	Experimental Physics II.	
Third.	Differential Equations.	Elementary Mechanics with graphic and numerical methods.	Introduction to Chemistry.	
Fourth.	Algebra with Theory of Numbers.	Curves and Surfaces.	Higher Mechanics.	History of Philosophy and Pedagogy.
Fifth.	Theory of Functions.	Geodesy with Probabilities.	Theoretic Physics I.	Logic.
Sixth.	Synoptic Course.	Astronomy with Geophysics.	Theoretic Physics II.	Psychology.

The report closes with a bibliography (pages 309–322) of 290 titles of publications appearing in the years 1900–07, inclusive, relative to the field of the commission's labors and classified as shown in the table on the next page.

Referring again to the earlier review in the BULLETIN of the important general proposals of the commission, it may be said in closing that this complete report presents in convenient

	General.	Mathematics.	Physics.	Chemistry, Geology, Biology.	Totals.
Books	11	16	5	6	38
Articles:					
Secondary Schools	61	79	14	40	194
Universities	14	28	7	9	58
Totals	86	123	26	55	290

form for permanent preservation and use the record of the entire activity of the commission, an activity which will not lose its significance for many years to come, and whose record will long remain a pedagogic document of the first importance.

J. W. A. YOUNG.

NOTES.

ALL the papers read at the joint meetings of mathematicians and engineers, held at Chicago in December, 1907, have been published in *Science*, and reprinted in the form of a 56-page pamphlet, for the use of the joint committee of fifteen, of which Professor E. V. HUNTINGTON is chairman, and which is to present its report, on the teaching of mathematics in colleges of engineering, at the summer meeting, 1909, of the Society for the promotion of engineering education. A limited number of these reprints are available for distribution among members of the AMERICAN MATHEMATICAL SOCIETY, on application to the Secretary of the Chicago Section, Professor H. E. SLAUGHT, 58th Street and Ellis Avenue, Chicago, Ill. They may be had in the order of application while they last.

THE opening (January) number of volume 10 of the *Transactions of the American Mathematical Society* contains the following papers: "Zur Differentialgeometrie der analytischen Curven," by E. STUDY; "The central of a group," by G. A. MILLER; "The hypergeometric functions of n variables," by J. I. HUTCHINSON; "Surfaces derived from the cubic variety having nine double points in four dimensional space," by V. SNYDER; "On a certain class of isothermic surfaces," by A. E. YOUNG; "A geometrical application of binary syzygies," by A. E. LANDRY; "Definite forms in a finite field," by L. E. DICKSON.

THE January number (volume 10, number 2) of the *Annals of Mathematics* contains: "Concerning a compound discontinuous solution of the surface of revolution of minimum area," by MARY E. SINCLAIR; "On the geometric properties of quartic curves possessing fourfold symmetry with respect to a point," by R. D. CARMICHAEL; "Extension of the sieve of Eratosthenes to arithmetical progressions and application," by J. C. MOREHEAD.

THE American association for the advancement of science will not hold a summer meeting this year, in order to avoid interference with the meeting of the British association at Winnipeg. The winter meeting will be held in Boston under the presidency of President D. S. JORDAN, of Stanford University. Professor E. W. BROWN is vice-president and Professor G. A. MILLER secretary of Section A.

THE second regular meeting of the Syracuse Section of the teachers of mathematics in the middle states and Maryland was held at the Syracuse high school, December 29, 1908. The following papers were presented. By H. DE W. GROAT, "Field notes on the teaching of secondary mathematics"; by A. M. CURTIS and C. E. BIKLÉ, "What should be the aims in teaching algebra, and how to attain them?"; by W. E. BOND and E. P. SISSON, "What should be the aims in teaching geometry and how to attain them?"

THE committee on organization of the International commission on the teaching of mathematics has appointed as the members of the commission to represent the United States Professors W. F. OSGOOD, D. E. SMITH, and J. W. A. YOUNG. The preliminary report, setting forth the general plan of the work, was published in *L'Enseignement Mathématique*, November 15, 1908.

FOLLOWING the amendment to the constitution enacted in August, 1907, the council of the Circolo matematico di Palermo now consists of five resident and forty non-resident members. Among the latter are Professor E. H. MOORE, of the University of Chicago, and Professor W. F. OSGOOD, of Harvard University.

AT the annual public meeting of the Paris academy of sciences, held on December 7, 1908, the following prizes were awarded for memoirs in pure and applied mathematics: Grand prize (fr. 3000) divided equally between Professor L. BIANCHI, of the University of Pisa, and Professor C. GUICHARD of the University of Clermont-Ferrand, for their memoirs on the deformations of quadric surfaces; Francœur prize (fr. 1000) to Dr. E. LEMOINE, for his contributions to geometry; Poncelet prize (fr. 2000) to Professor I. FREDHOLM, of the University of Stockholm, for his researches in functional analysis; Montyon prize (fr. 700) to E. LEBERT, chief engineer of the department of roads and bridges, for his contributions to mechanics; Fourneyron prize, not awarded; Guzman prize not awarded; Lalande prize (fr. 540) divided between Director W. L. ELKIN, of the Cape Town Observatory, and Mr. F. L. CHASE, of Yale University, with honorable mention of Mr. M. F. SMITH, of Yale University, for their investigations of stellar parallax; Valz prize (fr. 460) to Professor M. LUIZET, of the University of Lyons, for his researches in astronomy; Janssen prize (fr. 400) to Professor P. PUISEUX, of the University of Paris, Wilde prizes (two, fr. 2000 each) to Dr. M. TIKHOFF, of the Observatory of St. Petersburg, and Dr. C. NORDMANN, of the Observatory of Paris, for their researches on the dispersion of light and on variable stars; Houllevigue prize (fr. 500) was divided between Professor DEBIERNE, of the University of Paris, Professor PETOT, of the University of Lille, and Professor E. FABRY, of the University of Montpellier; Delcros prize (fr. 8000) to Professor J. HADAMARD, of the University of Paris; Laplace prize (complete works of Laplace) to Dr. P. M. E. LANCRENON, of the national school of mines.

The following programme of proposed prize subjects was announced. Grand prize (fr. 3000) (1910): "It is known how to find all the systems of two meromorphic functions of one complex variable, connected by an algebraic relation. An analogous problem is proposed for three uniform functions of two complex variables, having for finite regions the character of rational functions, and connected by an algebraic relation. In case a complete solution is not obtained, the academy demands at least some illustrative examples which lead to classes of new transcendental functions;" Francœur prize (fr. 1000) (1909) for work of merit in pure or applied mathematics; Poncelet prize (fr. 2000) in 1909 for a meritorious memoir in applied mathematics, in 1910 for a similar contribution to pure mathematics; Bordin prize (fr. 3000) (1911). "Complete in some important point, the theory of systems of triply orthogonal surfaces;" Montyon prize (fr. 700) (1910): "Invent or perfect useful machines in agriculture or the mechanic arts;" Vaillant prize (fr. 4000) (1911): "Complete, in some important point, the theory of the movement of an ellipsoid in an infinite, viscous liquid." Besides these, a number of prizes are offered for meritorious advances in any science.

THE philosophical faculty of the University of Göttingen announces the following problem, for the solution of which the faculty will award the Beneke prize in 1911:

"The number of oscillations which have been observed in the spectra of the elements seem to be distributed according to some definite law. They form so-called series. All the known observations should be collected and studied, and theories deduced from them should be critically examined. Moreover, further experiments are desired, to complete the known observations. It is probable that barium will show a three-fold series, corresponding to the known three-fold series of related substances."

Competing memoirs should be sent to the secretary before August 31, 1910. The result of the competition will be announced March 11, 1911. The value of the first prize is 3400 Marks, of the second is 680 Marks.

THE curators of the Wolfskehl foundation announce that Professor H. POINCARÉ has consented to deliver at Göttingen, April 22–28, a series of lectures on various mathematical subjects. Mathematicians generally are invited to attend these lectures. Sessions of the Göttingen mathematical society have also been arranged for the same week.

THE publishing house of B. G. Teubner in Leipzig and Berlin announces that the following mathematical works are in the press and will probably appear within a few weeks: "Encyklopädie," parts of volumes II_2, III_1, III_2, IV_2, V_2, V_3, VI_1, VI_2; "Encyclopédie," all the remaining parts of tome I, volumes 1, 2, 3, 4; part of tome II, volume 1, tome IV, volumes 2 and 3; "Grundlagen der Geometrie" (3d edition), by D. HILBERT; "Elementar-Mathematik vom höheren Standpunkte aus: Teil 2, Geometrie," by F. KLEIN; "Handbuch der Mathematik," by F. MEYER, H. THIEME, E. NETTO, and C. FÄRBER; "Einführung in die höhere Algebra," by M. Bôcher; "Grundzüge der Differential- und Integralrechnung," by G. Kowalewski; "Vorlesungen über Variationsrechnung" (part 2), by O. Bolza; "Projektive Geometrie der Ebene," by H. Grassmann; "Die Lehre von den geometrischen Verwandtschaften, Band III, Die eindeutigen linearen Verwandtschaften zwischen Gebilden dritter Stufe," by R. Sturm. The second installment of volume 1 in the second edition of Clebsch-Lindemann's "Vorlesungen über Geometrie," will also soon appear. The plan for the compilation of the second edition of volume 2 of Pascal's "Repertorium der höheren Mathematik" has now been completed. It is hoped that the German edition will appear before the end of the present year.

Professor M. Disteli, of the technical school at Dresden, has accepted a professorship of mathematics at the technical school of Karlsruhe.

Professor A. Sommerfeld, of the University of Munich, has been elected associate member of the Bavarian academy of sciences.

Professor Liapounoff, of the University of St. Petersburg, has been elected corresponding member of the Academia dei Lincei at Rome.

Dr. L. Kollros has been appointed docent in mathematics at the technical school of Zürich.

Dr. H. Hertzer, emeritus professor of mathematics at the technical school of Charlottenberg, died November 16, 1908, at the age of 78 years.

Dr. W. F. White, head of the department of mathematics in the State Normal School at New Paltz, N. Y., died November 29, 1908.

Second-hand catalogue: H. Hugendubel, 18 Salvatorstrasse Munich. Catalogue No. 39, containing 603 mathematical titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Arndt (B.). Ueber die Verallgemeinerung des Krümmungsbegriffes für Raumkurven. (Diss.) Königsberg, 1908. 8vo. 117 pp.

Brandes (H.). Ueber die axiomatische Einfachheit mit besonderer Berücksichtigung der auf Addition beruhenden Zerlegungsbeweise des Pythagoräischen Lehrsatzes. (Diss.) Halle, 1908. 8vo. 50 pp.

Burkhardt (H.). Funktionentheoretische Vorlesungen. Vol. I. 1tes Heft: Algebraische Analysis. 2te, durchgesehene und vermehrte Auflage. Leipzig, Veit, 1908. 8vo. 12 + 199 pp. Cloth. M. 6.60

Buzengeiger (K.). See Feuerbach (K. W.).

Carus (P.). The foundations of mathematics: a contribution to the philosophy of geometry. Chicago, Open Court Pub. Co., 1908. 12mo. 4 + 141 pp. Cloth. $0.75

Darboux (G.). See Fabry (E.).

Dieck (W.). Zur Klassifikation der Punktepaar- und Kegelschnitt-Büschel. (Diss.) Kiel, 1908. 8vo. 40 pp.

Erfurth (P.). Die Komplementärflächen der pseudosphärischen Schraubenflächen. (Diss.) Halle, 1908. 8vo. 59 pp.

Fabry (E.). Traité de mathématiques générales à l'usage des chimistes, physiciens, ingénieurs et des élèves des facultés des sciences. Avec une préface de G. Darboux. Paris, Hermann, 1909. 8vo. 10 + 440 pp. Fr. 9.00

Feuerbach (K. W.). Eigenschaften einiger merkwürdigen Punkte des geradlinigen Dreiecks und mehrerer durch sie bestimmten Linien und Figuren. Eine analytisch-trigonometrische Abhandlung. Mit einer Vorrede von K. Buzengeiger. 2te nichtgeänderte Ausgabe. Haarlem, 1908. 8vo. 13 + 73 + 6 pp. M. 2.25

Filon (L. N. G.). An introduction to projective geometry. London, Arnold, 1908. 8vo. 258 pp. Cloth. 7s. 6d.

Heiland (F.). Hüllflächen einer Schar von Regelflächen zweiter Ordnung. (Diss.) Jena, 1908. 8vo. 69 pp.

Heinemann (E.). Die gemischte Kegelschnittschar [Das Kegelschnittsystem S (1 p, 3 l)]. (Diss.) Giessen, 1908. 8vo. 84 pp.

Hellinger (E.). See Klein (F.).

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LIBRARY.

THE FIFTEENTH ANNUAL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

SINCE the founding of the Society in 1888, the regular, including the annual, meetings have been held almost without exception in New York City, as the most convenient center for the members living in the eastern states and others who might from time to time attend an eastern meeting. The summer meeting, migratory between limits as far apart as Boston and St. Louis, has afforded an annual opportunity for a fully representative gathering, and provision has been made for the convenience of the central and western members by the founding of the Chicago Section in 1897, the San Francisco Section in 1902, and the Southwestern Section in 1906. The desire has, however, often been expressed that the annual meeting of the Society might, when geographic and other conditions were exceptionally favorable, be occasionally held like that of many other scientific bodies in connection with the meeting of the American association for the advancement of science, a gathering which naturally affords many conveniences of travel and scientific advantages. It was therefore decided to hold the annual meeting of 1908 at Baltimore in affiliation with the Association, the days chosen being Wednesday and Thursday, December 30–31.

Two sessions were held on each day in the Biological Laboratory of Johns Hopkins University. The total attendance numbered about seventy-five, including the following fifty-seven members of the Society:

Miss C. C. Barnum, Dr. E. G. Bill, Professor G. A. Bliss, Professor E. W. Brown, Dr. A. B. Chace, Professor A. S. Chessin, Dr. A. B. Coble, Dr. Abraham Cohen, Professor F. N. Cole, Dr. G. M. Conwell, Miss E. B. Cowley, Mr. F. F. Decker, Mr. C. E. Dimick, Professor John Eiesland, Professor L. P. Eisenhart, Professor J. C. Fields, Dr. Fabian Franklin, Dr. W. A. Granville, Dr. F. L. Griffin, Professor C. C. Grove, Professor Harris Hancock, Professor J. G. Hardy, Professor C. N. Haskins, Rev. A. S. Hawkesworth, Dr. A. A. Himowich, Professor H. L. Hodgkins, Professor L. S. Hulbert, Professor J. I. Hutchinson, Professor Edward Kasner, Professor C. J. Keyser, Mr. W. J. King, Mr. Joseph Lipke, Dr. W. R. Longley, Professor W. H. Maltbie, Professor Mansfield Merriman, Professor J. F. Messick, Professor G. A. Miller, Professor J. A. Miller, Dr. C. L. E. Moore, Professor Frank Morley, Professor G. D. Olds, Professor N. A. Patillo, Dr. H. B. Phillips, Professor E. D. Roe, Professor F. H. Safford, Miss M. E. Sinclair, Professor P. F. Smith, Professor Virgil Snyder, Professor H. F. Stecker, Professor J. H. Tanner, Professor E. J. Townsend, Professor H. W. Tyler, Professor Anna L. Van Benschoten, Professor C. A. Waldo, Professor H. S. White, Professor T. W. D. Worthen, Professor J. E. Wright.

Wednesday evening was set apart for a social reunion and dinner, at which forty-five members were present. The meeting of the Council was held later on the same evening.

The annual meeting was especially marked as the occasion of the retiring address of President H. S. White on "Bezout's theory of resultants and its influence on geometry," which was delivered at the opening of the afternoon session on Wednesday.

The heavy programme fully occupied the four sessions. President White occupied the chair, being relieved by Professor Morley and Vice-Presidents G. A. Miller and Kasner. The Council announced the election of the following persons to membership in the Society: Professor G. N. Armstrong, Ohio Wesleyan University; Professor P. F. Gaehr, Robert College, Constantinople; Dr. Frank Irwin, Princeton University; Miss Mary E. Wells, Mount Holyoke College. Six applications for membership were received.

The reports of the Treasurer, Auditing Committee, and Librarian will appear in the Annual Register, now in press. The membership of the Society has increased during the past year from 582 to 601, including at present 55 life members. The number of papers presented at all meetings during the year was 169. The total attendance of members at the meetings was 281. The Treasurer's report shows a balance of $6995.80, of which $3215.20 is credited to the life membership fund. Sales of the Society's publications during the year amounted to $1385.36.

At the annual election, which closed on Thursday morning, the following officers and members of the Council were chosen:

President,	Professor MAXIME BÔCHER.
Vice-Presidents,	Professor EDWARD KASNER.
	Professor E. B. VAN VLECK.

Secretary, Professor F. N. COLE.
Treasurer, Professor J. H. TANNER.
Librarian, Professor D. E. SMITH.

Committee of Publication,
Professor F. N. COLE,
Professor D. E. SMITH,
Professor VIRGIL SNYDER.

Members of the Council to serve until December, 1911,
Professor H. B. FINE, Professor F. R. MOULTON,
Professor O. D. KELLOGG, Professor E. J. WILCZYNSKI.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On r-fold symmetry of plane algebraic curves."

(2) Professor R. D. CARMICHAEL: "A general principle of inversion, with applications."

(3) Dr. W. R. LONGLEY: "Some sufficient conditions in the theory of implicit functions."

(4) Dr. C. L. E. MOORE: "Properties of systems of lines in space of four dimensions and their interpretation in circle geometry."

(5) Dr. F. R. SHARPE: "The topography of the integral curves of a differential equation."

(6) Mr. JOSEPH LIPKE: "Note on isotropic ruled surfaces."

(7) Professor JOHN EIESLAND: "On a species of cubic surfaces of the sixth class."

(8) President E. O. LOVETT: "Integrable problems of three bodies."

(9) Professor J. I. HUTCHINSON: "On linear transformations which leave an hermitian form invariant."

(10) President H. S. WHITE: — Presidential Address: "Bezout's theory of resultants and its influence on geometry."

(11) Professor FRANK MORLEY: "Plane sections of the Weddle surface."

(12) Professor G. A. MILLER: "Finite groups which may be defined by two operators satisfying two conditions."

(13) Dr. F. L. GRIFFIN: "Tests comparing the apsidal angles and periodic times for different laws of central force."

(14) Dr. E. G. BILL: "Existence 'im Kleinen' of a space curve which minimizes a definite integral."

(15) Dr. E. G. Bill: "An a priori existence theorem in three dimensions for the calculus of variations."

(16) Mr. J. R. Conner: "Curves and surfaces which admit configurations of the Cayley-Veronese type."

(17) Mr. D. D. Leib: "The complete system of invariants for two triangles."

(18) Dr. W. A. Granville: "Dual formulas in spherical trigonometry."

(19) Professor C. J. Keyser: "Concerning euclidean geometries without points and lines."

(20) Miss M. E. Sinclair: "The problem of the surface of revolution with two end points variable on circles."

(21) Professor G. A. Bliss: "On the construction of the coordinate system of analytic projective geometry."

(22) Professor Edward Kasner: "The group generated by turns and slides."

(23) Professor Edward Kasner: "Catenaries in an arbitrary field of force."

(24) Professor C. N. Haskins: "Numerical computation of reaction velocity constants."

(25) Dr. W. B. Carver: "Degenerate pencils of quadrics connected with $\Gamma^{n+2}_{n+4,n}$ configurations."

(26) Dr. C. F. Craig: "On a class of hyperfuchsian functions."

(27) Professor Virgil Snyder: "Surfaces and congruences derived from the cubic variety having a double line in four-dimensional space."

(28) Professor W. B. Fite: "Irreducible homogeneous linear groups in an arbitrary infinite field."

(29) Dr. Arthur Ranum: "On certain solids in riemannian space."

(30) Dr. Arthur Ranum: "On the rank of a matrix."

(31) Dr. H. B. Phillips: "Polygons on a quadric surface."

(32) Professor J. G. Hardy: "Note on a theorem of Pirondini concerning four-dimensional curves."

(33) Dr. A. B. Coble: "Combinants of binary forms."

Mr. Conner and Mr. Leib were introduced by Professor Morley. In the absence of the authors the papers of Dr. Sharpe, Dr. Carver, Dr. Craig, Professor Fite, and Dr. Ranum were read by Professor Snyder, and the papers of Professor Carmichael, President Lovett, and Dr. Phillips were read by title. President White's address will appear in the next number of the BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. This paper is a generalization of a preceding one on fourfold symmetry of plane algebraic curves. Professor Carmichael shows that there are still two classes of curves distinguished by differing forms of the equation. Taking the general cartesian equation in the form

$$\sum a_{ts} x^t y^s = 0 \qquad (t + s \leqq n),$$

with origin at the center of symmetry, it is shown that for r-fold symmetry the necessary and sufficient relations among the constants a in the two classes are respectively

$$\sum_{t,\,s=0}^{t,\,s=v} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = \sum_{t,\,s=0}^{t,\,s=v} a_{ts} \cos^t (\theta_1 + \alpha\phi) \sin^s (\theta_1 + \alpha\phi), \qquad (t + s = v),$$

$$\sum_{t,\,s=0}^{t,\,s=v} a_{ts} \cos^t \theta_1 \sin^s \theta_1 = (-1)^a \sum_{t,\,s=0}^{t,\,s=v} a_{ts} \cos^t (\theta_1 + \alpha\phi) \sin^s (\theta_1 + \alpha\phi), \qquad (t + s \neq v),$$

where $v = 0, 1, 2 \cdots, n$ (different equations being formed for each value of v), α is any integer $\leqq r - 1$, and $\phi = 360^\circ / r$. Methods are given for determining the a's in each case and the results are worked out for the smaller values of r and n. Finally, an example is given of a special tenfold symmetrical curve.

2. G. Cantor has developed several properties of the function $\beta(N)$ defined by

$$\beta(1) = 1, \quad \beta(N) = (-1)^{a_1 + a_2 + \cdots + a_n},$$

where $N = p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n} > 1$, $p_1, p_2, p_3, \cdots, p_n$ being different primes. Professor Carmichael establishes the following inversion principle:

If n is an integer and $f(n)$ and $F(n)$ are two functions connected by the relation $f(n) = \sum_k F(kn)$, but otherwise quite arbitrary, then $\sum_n F(n^2) = \sum_n \beta(n) \cdot f(n)$, where the summation may extend to an infinite number of terms or may stop short at a finite number of terms by the assumption that $f(n)$ and $F(n)$ have only the value zero when n is greater than some fixed number.

Several applications are made of this principle. In particular it is applied to the summation of certain classes of infinite series.

3. In Dr. Longley's paper the functions considered are real functions of real variables. The following theorem on implicit functions is well known: Consider the system of equations

$$(1) \qquad f_i(x_1, \cdots, x_n;\quad u_1, \cdots, u_m) = 0 \quad (i = 1, \cdots n).$$

Suppose a special solution

$$(2) \qquad x_1 = a_1, \cdots, x_n = a_n;\quad u_1 = b_1, \cdots, u_m = b_m,$$

where a_i and b_i denote constants, is known, and that in the neighborhood of the system of values (2) the functions f_i are of class* C'. Suppose the functional determinant

$$\Delta = \frac{\partial(f_1, \cdots, f_n)}{\partial(x_1, \cdots, x_n)} \neq 0$$

for the system of values (2). Then equations (1) have one and only one solution of the form

$$A: \qquad x_i = \phi_i(u_1, \cdots, u_m),$$

where $a_i = \phi_i(b_1, \cdots, b_m)$.

The hypothesis of this theorem states a condition which is sufficient to insure the existence of a unique solution of the type A. The condition $\Delta \neq 0$ is however not necessary. Supposing that $\Delta = 0$, it is the object of this paper to present some other conditions which are sufficient to insure the existence of a unique solution of the type A. The geometric significance of some of the theorems is considered. Applications are made to the existence, in the plane and in space, of a field of extremals about a point where all the extremals of the set pass through that point.

4. If the equations of a line in a space of four dimensions S_4 be written in the form

$$x_1 = a_1 + \alpha_1 z, \quad x_2 = a_2 + \alpha_2 z, \quad x_3 = a_3 + \alpha_3 z, \quad x_4 = z,$$

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* That is, continuous with continuous first partial derivatives.

then the equations of the tangent plane to a ruled surface in S_4 may be written

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_2 - a_2 - \alpha_2 x_4} = \frac{a_1' + \alpha_1' z}{a_2' + \alpha_3' z} = \rho_1,$$

$$\frac{x_1 - a_1 - \alpha_1 x_4}{x_3 - a_3 - \alpha_3 x_4} = \frac{a_1' + \alpha_1' z}{a_3' + \alpha_3' z} = \rho_2 .$$

Considering a_1', a_2', a_3', α_1', α_2', α_3' as the homogeneous coordinates of the projectivity between the points z of any generator and the planes passing through the generator and tangent to the ruled surface, Dr. Moore discussed the general properties of lines in space of four dimensions and interpreted the results in circle geometry in space of three dimensions.

5. Dr. Sharpe's paper discusses the topography of the integral curves of $dy/dx = c_1/c_2$ where $c_1 = 0$, $c_2 = 0$ are conics. The nature of the integral curves depends on the locus of points of the conic $c_1 = c_2\lambda$ at which the conic has the slope λ. This locus is also the locus of the inflections of the integral curves and is a quintic having a double point at each of the four intersections of $c_1 = 0$, $c_2 = 0$ and passing through the three intersections of the three pairs of lines through the four points. When the conics $c_1 = 0$, $c_2 = 0$ have simple contact, the quintic has a triple point at the point of contact. When the contact is of the second order, the triple point consists of a cusp and a simple branch. When the contact is of the third order, the quintic degenerates into the common tangent and a quartic having a tacnode at the point of contact. Examples are given of the most interesting cases.

6. In discussing the problem of the line of striction on a ruled skew surface, it is generally shown that an indeterminacy arises if the generator is a minimal straight line. In this note, Mr. Lipke defines the line of striction in one of the usual ways and, introducing infinitesimals of the second order, gets a definite curve on the isotropic skew surface as its line of striction. He further discusses the relations existing between this curve, the edge of regression of the minimal developable enveloped by the system of minimal planes through the generators, and the focal curve of the congruence of normals to the surface.

7. Professor Eiesland's paper is a study of cubic surfaces of the form

$$A + Bx + Dy + Cz + Exz + Fxy + Gyz + Hxyz = 0,$$

with the identical relation $EGAF = HDCB$ between the coefficients. It is shown that two pairs of families of twisted cubics exist on it such that by the involutory transformation

$$\bar{x} = \frac{\lambda}{x}, \quad \bar{y} = \frac{\mu}{y}, \quad \bar{z} = \frac{\nu}{z},$$

where λ, μ, ν are properly chosen values, each pair is transformed into the other. These surfaces are connected with a unicursal quartic in the plane at infinity.

8. Generalizing a well-known problem of Bertrand in mechanics, Oppenheim showed in 1894* that if three given particles describe under central conservative forces three given plane curves, the forces are derived from a potential function which may be written in a form in which there appear only the constant of areas, the masses of the bodies, their coordinates referred to the center of gravity of the system, and the first partial derivatives of the functions defining the orbits. Oppenheim applied his results to the rediscovery of the known solutions of the plane problem of three bodies in which the law of force is as the masses, and either directly as the distance or inversely as the square of the distance.

In Professor Lovett's paper, by the aid of the theory of partial differential equations of the first order and the form of the force function revealed in Oppenheim's memoir, there is constructed an infinitude of integrable problems of three bodies in the plane under forces depending on the masses and mutual distances of the bodies. These problems are grouped in four families according to the form of the potential function. The paper concludes with a fifth family in which the coordinates of the bodies appear explicitly in the force function.

9. The object of the paper by Professor Hutchinson is to give some of the properties of linear transformations on n variables for which an hermitian form H is invariant. The substitutions whose characteristic determinants have no equal roots are classified into elliptic (all roots unimodular) or hyperbolic of q differential types (when $2q$ of the roots are not unimodular).

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* *Publicationen der v. Kuffner'schen Sternwarte*, vol. 3, pp. 71–95.

The fixed points of an elliptic substitution are situated so that p of them are inside and $n - p$ outside the locus $H = 0$ (p being the number of positive terms of H and $p \leqq n - p$). The fixed points for a hyperbolic substitution of the qth type include q pairs which are on H, $p - q$ points inside H, and $n - p - q$ outside H ($q \leqq p$). The generalized Poincaré series is considered, and the form of its expansion in the vicinity of a fixed point determined.

11. The plane section of a Weddle surface is not an arbitrary quartic curve, but one for which an invariant vanishes. The curve contains a configuration B^6, namely, where it is cut by the lines on the surface. The theorem proved by Professor Morley is that it contains an infinity of such configurations.

12. It is evident that two or more distinct operators, none of which is identity, cannot be sufficiently restricted by a single condition of the form $s_1^\alpha s_2^\beta s_1^\gamma s_2^\delta \cdots = 1$, where $\alpha, \beta, \gamma, \delta, \cdots$ are positive or negative integers, to generate only groups whose orders are limited by the values of $\alpha, \beta, \gamma, \delta, \cdots$. On the other hand it is possible to find pairs of such conditions which are such that each pair is satisfied by two generators of only a limited number of groups. Such a pair cannot be of the form $s_1^\alpha = 1$, $s_1^\beta = 1$ since the order of the product of two operators is not limited by the orders of these operators. In the present paper Professor Miller considers only those cases where a pair of conditions between two operators s_1, s_2 determines either only a single group or a limited number of groups.

If the two conditions are of the form $s_1^n = 1$, $s_1^\alpha = s_2^\beta$ the order of G is limited when n and α are relatively prime, and only then. When these conditions are satisfied, G is cyclic and its order divides βn. For instance, the cyclic group of order 40 is completely defined by the two conditions $s_1^{10} = 1$, $s_1^3 = s_2^4$. On the other hand, the two conditions $s_1^{10} = 1$, $s_1^4 = s_2^3$ are satisfied by two generators of an infinite number of distinct groups. The following theorem is useful in the study of some of the pairs of conditions: If two commutative operators satisfy the condition $s_1^n = s_2^m$, they generate the direct product of two cyclic groups whose orders are respectively the lowest common multiple of the orders of these operators, and a divisor of the highest common factor of α, β. The latter group may be identity, and it must be identity whenever α, β are relatively prime.

A large number of interesting results are obtained when the two conditions are of the form $s_1^n = 1$, $s_1 s_2 = s_2^\alpha s_1^\beta$. Among these are the following: These two conditions are satisfied by the two generators of only a finite number of groups when $\beta = 1$ and $\alpha \neq 1$, but they are satisfied by two generators of each one of an infinite system of distinct groups when also $\alpha = 1$. If two generators of a group satisfy the conditions $s_1^3 = 1$, $s_1 s_2 = s_2^2 s_1^{-1}$, the group must be either tetrahedral or the group of order 24 involving no subgroup of order 12.

13. In an earlier paper * Dr. Griffin proved certain theorems concerning the apsidal angle Θ in central orbits, obtaining, for example, conditions upon any given law of force necessary and sufficient in order that $\Theta > \pi$ in all orbits of any region. An examination of the method of proof, which consisted of a comparison of the definite integrals giving Θ and π (the corresponding angle for Newton's law), suggests the possibility of generalizing the theorems so as to compare the values of Θ in orbits under any two arbitrarily chosen laws of force.

Several such generalizations are given in the present paper, and applications are made to various simple laws of force. The new proofs, while more involved than those of the former paper, proceed along the same general lines. The criteria again are very simple and easily applied.

Analogous theorems are established for the periodic times in the orbits.

14. Hilbert in 1899 stated that under certain conditions a solution of the calculus of variations problem always existed, and later sketched a proof of this statement which Professor Bolza has since completed.† In this proof the integrand of

$$\int_{t_0}^{t_1} F(x, y, x', y') dt$$

is defined only for a finite closed region and the minimizing curve is built up as the limiting position of an "infinite" sequence of division points of a set of curves which approximate to giving the integral its minimum value.

In Dr. Bill's papers the similar question is considered for the integral

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* Cf BULLETIN, vol. 14 (Oct., 1907), pp. 6–16.

† Bolza, Lectures on the Calculus of Variations, p. 245.

$$J = \int_{t_0}^{t_1} F(x, y, z, x', y', z')dt,$$

carried over a curve in space.

In his first paper it is shown that a solution "im Kleinen" exists, the proof depending on the existence of a field of extremals through a given point.

15. In his second paper Dr. Bill proves the existence "im Grossen," by a method whereby the minimizing curve is obtained by applying the existence theorem "im Kleinen" to a "finite" number of points which are defined as limiting points of points lying on a sequence of approximating curves. Considerable simplification over the original form of the method of Hilbert is thus obtained.

16. Mr. Conner's paper is in abstract as follows: In a space of four dimensions, lines bisecant to a curve C_p^m, of order m and genus p, lie on a 3-way spread Λ of order $h = \frac{1}{2}(m-2)(m-3)-p$, in which C_p^m is an $(m-2)$-fold curve. A space α meets Λ in a surface K_h^m of order h. Denote by Δ_m and Γ_σ, respectively, the plane section of a complete m-point in three dimensions, and the space section of a complete σ-point in four dimensions. σ points on C_p^n define a configuration Γ_σ inscribed in K_h^m. A plane section of Λ admits Δ_m's determined by the sets of m points in which spaces on the plane meet C_p^m. C_0^4 gives rise to a 4-nodal cubic surface K_3^4. The theory of the K_3^4 is connected with that of the rational space quartic curve, the underlying quartic being here the (class) quartic in which the tangential developable to C_0^4 meets α. The surface K_5^5 is the general quintic surface with five triple points. The surface K_6^5 has besides five triple points a double curve of order 7. Curves C_p^m in five dimensions determine in a similar way space curves admitting Γ_m's.

17. In this paper Mr. Leib derives a complete system of the invariants of two triangles, and proves its completeness. Various relations are discussed, including the condition on two triangles so that a conic may be inscribed in one and circumscribed about the other. This leads at once to the equation of the locus of the foci of parabolas through three points, and properties of this locus are deduced directly from the invariant form. The covariants of special interest are also discussed.

18. As generally given by writers on spherical trigonometry, the formulas for the sides of a spherical triangle in terms of the angles have had a certain resemblance to the formulas giving the angles in terms of the sides, but the correspondence has not been perfect. Möbius was the first to point out the fact that if the supplements of the angles of the triangle were used, instead of the angles themselves, one set of these formulas is the dual of another. He did not, however, publish any such sets of formulas, and in this paper Dr. Granville shows what the ordinary formulas of spherical trigonometry become when given in this form. The following theorem is first established:

In any relation between the parts of a general spherical triangle, each part may be replaced by the supplement of the opposite part, and the relation thus obtained will hold true.

When the above theorem is applied to a relation involving one or more of the sides and the supplements of the angles of a spherical triangle, we get the following principle of duality:

If the sides of a general spherical triangle are denoted by the Roman letters a, b, c, and the supplements of the corresponding opposite angles by the Greek letters α, β, γ, then from any given formula involving any of these six parts, we may write a dual formula by simply interchanging the corresponding Greek and Roman letters. The old and the new sets of formulas are then compared, and spherical triangles are solved using both.

19. A system of postulates equivalent to the system (H) of Hilbert's Foundations or to Veblen's system (V) of geometric axioms (*Transactions*, volume 5) — happily described by the latter as categorical — define, not a class s of objects satisfying the system, but a class or category C of such classes s. C is determined by various systems, and is composed of manifold classes s. To each s corresponds a geometric theory determined by any one of the systems and the same for all of them. These theories, though one logically, are many psychologically. Hence the interest attaching to study directed upon the varied content as distinguished from the definition of C. If e, e', e'' denote the undefined elements of (H), e will represent the undefined element of (V), e' and e'' being in the latter system defined by means of the undefined notion of order. The problem is to find near-lying interpretations of the elements, *i. e.*, to discover such components s of C as are easily accessible and

hence interesting to intuition. After the usual interpretation, the most familiar one is that in which *s* is ordinary space bereft of a point *O*, *e* means point, and *e′* and *e″* denote respectively the circles and spheres that would contain *O* if it were not excluded. In Professor Keyser's paper the following classes *s* were indicated: (1) *s* is ordinary space (of planes), lacking any chosen sheaf or bundle *o* of planes, *e* denotes ordinary plane, *e′* denotes pencil of planes lacking the plane belonging to *o*, and *e″* denotes sheaf of planes lacking the *e′* belonging to *o*; (2) *s* is the same as before, *e* denotes sphere containing the vertex *P* of *o*, *e′* denotes sphere range (or circle) determined by two *e*'s, and *e″* denotes sphere congruence (or circle congruence) determined by three *e*'s not belonging to a same *e′*. The (*H*) and (*V*) axioms for two dimensions are satisfied by the following: (1) *s* is the ordinary plane of lines lacking any chosen pencil *o*, *e* denotes line, and *e′* denotes pencil lacking the *e* of *o*; (2) *s* is same as before, *e* is circle through the vertex *P* of *o*, and *e′* is circle range determined by two *e*'s; (3) *s* is a sheaf of planes minus a pencil *o*, *e* is plane, and *e′* is pencil of planes minus the *e* belonging to *o*; (4) *s* is a sheaf of lines lacking a pencil *o*, *e* is line, and *e′* is pencil of lines minus the *e* of *o*. Another *s* is a sheaf of circular cones of common vertex minus a pencil of them.

20. Miss Sinclair's paper is in abstract as follows: The original Plateau experiment of a catenoid film between two anchor rings is the problem of the surface of revolution of minimum area in case the two end points are variable on two circles equal and equidistant from the axis of revolution. Let the x-axis be the axis of revolution, and the y-axis pass through the center of the first circle. Let ρ be the radius of the circles, d the distance of their centers from the axis, (κ, a) the vertex of the catenary. Then we have the equations

$$(1)\qquad x_0 = \rho \cos \theta, \quad y_0 - d = \sin \theta;$$

$$(2)\qquad y_0 = aC(u_0), \quad x_0 = au_0 + \kappa;$$

$$(3)\qquad 1 - \frac{x_0}{y_0 - d} S(u_0) = 0,$$

which are sufficient to determine x_0, y_0, a, κ if u_0 is known. On account of the symmetry of the problem, x_1, y_1 can be found from (2) by replacing u_0 by $-u_0$.

The parameter u_0'' of the point focal to u_0 is the positive solution of

$$(4)\qquad \frac{\rho}{a} = C^2(u_0)S(u_0)\frac{\phi(u_0'') - \phi(u_0)}{\phi(u_0'') + u_0 + C(u_0)S(u_0)},$$

where

$$\phi(u_0) = \frac{C(u_0)}{S(u_0)} - u_0.$$

The value of u_0 is restricted by the conditions for fixed end points, on account of the symmetry of the problem, to the interval $(u_0^*, 0)$ where u_0^* is the negative solution of $\phi(u_0) = 0$. The Bliss condition for two variable end points requires for one problem that $\phi(u_0'') \leqq 0$. For the particular value u_0'' for which $\phi(u_0'') = 0$ we obtain, from the above condition,

$$(5)\qquad \frac{\rho}{d} = \frac{-C(u_0)\,S(u_0)\,\phi(u_0)}{[1 - S(u_0)]u_0 + 2C(u_0)S(u_0)}.$$

Moreover, ρ/d is a continuous function of u_0 in the interval $(u_0^*, 0)$, increasing from 0 to ∞. Hence if ρ and d are given, (5) may on this interval be solved uniquely for u_0. Let this solution be $\bar{u}_0$. If ρ and d are constant, u_0'' is found to be an increasing function of u_0, and the Bliss condition is satisfied for $u_0 > \bar{u}_0$.

A physical illustration is of interest and the results of experiment are here given:

$$\rho = 2.5 \text{ cm.},\quad \theta = 33^\circ,$$

$$d = 5 \text{ cm.},\quad \text{length of film} = 2\kappa = 3.08.$$

The corresponding theoretical results are (ρ and d being given)

$$\rho = 2.5 \text{ cm.},\quad \theta = 33^\circ\,0'\,30'',$$

$$d = 5 \text{ cm.},\quad 2\kappa = 3.01.$$

21. In Professor Bliss's paper it is shown how the projective coordinates of points and lines in the plane may be defined in terms of anharmonic ratios, and how the relation

$$u_1x_1 + u_2x_2 + u_3x_3 = 0$$

expressing the incidence of the point (x_1, x_2, x_3) and the line (u_1, u_2, u_3) may be proved directly from the properties of anharmonic ratios. The definition of the coordinates is not new, but the proof of the incidence relation seems more direct than those usually given.

22. The transformations considered in Professor Kasner's first paper operate on the oriented lineal elements of the plane. By a turn T_α each element rotates about its own point through a fixed angle α. By a slide S_κ each element moves along its own line through a fixed distance κ. It is shown that every combination of turns and slides is reducible to the form $T_\alpha S_\kappa T_\beta$, so that a continuous group of three parameters is obtained. The only contact transformations included are the dilatations. Applied to a simply infinite system of curves the turns lead to the familiar isogonals and the slides to the equitangentials whose theory has been developed recently by Scheffers. The combination of both constructions leads to ∞^3 related systems (though in special cases, admitting automorphic transformations, there will be fewer). The two known theories of trajectories, which appear isolated in the usual treatment, are shown to be special cases of one general theory. The fundamental figure considered, which includes the point, line, and circle as special cases, is formed of ∞^1 elements whose points are on a circle and whose directions are equally inclined to that circle. It is termed a geometric *turbine*. The most general transformation of turbines involves 15 parameters. The resulting geometry is isomorphic with projective geometry in three dimensions. Many previous theories appear in a new light.

23. The study of the equilibrium of a homogeneous flexible inextensible string in a given field of force leads to ∞^6 curves in space, or ∞^3 in the plane. These general systems of catenaries are explored in Professor Kasner's second paper. The results are analogous to but distinct from those dealing with dynamical trajectories (*Transactions*, 1906, 1907). Characteristic properties are obtained, largely by intrinsic methods. It is shown that the field of force may be constructed when a sufficient (infinite) number of its catenaries are known. The modifications necessary for non-homogeneous strings are considered.

24. The paper of Professor Haskins has reference to a problem of chemical dynamics. The course of an incomplete bimolecular reaction is determined by the differential equation

$$dx/dt = k_1(a_1x^2 + 2b_1x + c_1) - k_2(a_2x^2 + 2b_2x + c_2),$$

where a_1, b_1, c_1, a_2, b_2, c_2 are constants known from the initial conditions, and k_1, k_2 are the so-called "reaction constants" which are to be determined from simultaneous observations of x and t. In the reactions hitherto studied the ratio k_1/k_2 has been found from the equilibrium conditions ($t = \infty$) and in this case the computation of k_1/k_2 offers no difficulty. In recent investigations of gas reactions the equilibrium conditions were not attainable, and a method of determination of k_1, k_2 from any two pairs of values of x and t became necessary. The determination can be reduced to the solution of the equation in γ

$$\log\frac{\gamma + z_2}{\gamma - z_2} - \frac{t_2}{t_1}\log\frac{\gamma + z_1}{\gamma - z_1} + \left(\frac{t_2}{t_1} - 1\right)\log\frac{\gamma + z_0}{\gamma - z_0} = 0,$$

$$(z_0 < z_1 < z_2, \quad t_1 < t_2),$$

and in important special cases to that of

$$\log\frac{\gamma + z_2}{\gamma - z_2} - \frac{t_2}{t_1}\log\frac{\gamma + z_1}{\gamma - z_1} = 0, \quad (z_1 < z_2, \quad t_1 < t_2).$$

A single condition must be satisfied in order that a root may exist, and the root if existent is unique. The numerical solution proceeds by iteration and converges in general with considerable rapidity.

25. In a paper presented to the Society at the December meeting, 1906 (which has since appeared in the January number of the *American Journal of Mathematics*), Dr. Carver called attention to certain pencils of quadrics connected with the configuration $\Gamma^{n+2}_{n+4,n}$. The purpose of the present note is to investigate the question of the existence of degenerate pencils for configurations which are not degenerate.

26. Alezais (*Annales de l'Ecole normale supérieure*, (3), volume 19 (1902), page 261) has studied a monodromy group of a particular Riemann surface. Dr. Craig's paper studies a simpler group and functions belonging to the group. An extension of a known result on the number of linearly independent moduli belonging to an abelian integral of the first kind on a

binomial Riemann surface, and the upper limit for the number of generators of the maximum monodromy group connected with such surfaces are obtained.

27. By the same methods as were employed in the study of cubic varieties having nine double points (abstract in the BULLETIN, volume 15, page 67) Professor Snyder examines all the forms having one or more double lines, filling in the details outlined by the Segre memoirs and supplying a large number of particular cases. All the particularizations of the (3, 2) congruence having a focal line can be obtained and classified from this standpoint, and also the forms of the (2, 2) congruences, the latter being already known. Finally, a large number o surfaces are obtained which remain invariant under birational transformations of infinite order.

28. The principal result in Professor Fite's paper is the following theorem: A necessary and sufficient condition that any group of finite order be simply isomorphic with an irreducible group in any infinite field is that its central be cyclic (or identity). As a direct application of this theorem, the number of those substitutions of finite order that are commutative with a given substitution that is irreducible in any infinite field is determined.

29. Defining a parallelogram in riemannian space as a portion of a Clifford surface bounded by a skew quadrilateral whose opposite sides are right parallels and left parallels, respectively, and a parallelepiped as a solid bounded by three pairs of opposite faces, all of which are parallelograms, Dr. Ranum finds that there are two distinct species of parallelepipeds, one depending on six parameters and the other on two parameters. Defining a quasi-parallelogram as a plane quadrilateral whose opposite sides are congruent, and a quasi-parallelepiped in a corresponding manner, he shows that while there exist oblique quasi-parallelograms (having no lines of symmetry), nevertheless oblique quasi-parallelepipeds (having no planes of symmetry) do not exist.

30. In order to determine the rank of a matrix it is sometimes of practical utility to know the minimum number of determinants of a given order which must be examined as to their vanishing or non-vanishing. In this paper Dr. Ranum con-

siders a matrix having m rows and n columns, and under certain hypotheses as to the non-vanishing of one or more r-rowed determinants of the matrix finds the minimum number of s-rowed determinants ($s > r$) whose vanishing will insure the vanishing of all the other s-rowed determinants.

31. The polygons of Dr. Phillips's paper are formed of groups of $2n$ points joined in order by generators of a quadric surface. From a remark of Salmon it follows that if each odd point is joined to the non-adjacent even points the $n(n-2)$ lines lie on a surface of class $n-2$. The object of the paper is a discussion of these surfaces.

32. Professor Hardy's paper takes up a theorem of Pirondini concerning "cylindrical helices" in a space of four dimensions and shows that under Pirondini's assumptions such curves lie entirely in a properly chosen three dimensional space, and are not, properly speaking, four-dimensional curves.

33. The involution $I^n_{\kappa, n-\kappa}$ of ∞^κ sets of n values such that κ values in a set determine the remaining $n-\kappa$ is expressed analytically by a *special* symmetric binary form in $\kappa+1$ variables to the order $n-\kappa$,

$$I_{\kappa, n-\kappa} \equiv (a_1x_1)^{n-\kappa}(a_2x_2)^{n-\kappa}\cdots(a_\kappa x_\kappa)^{n-\kappa}(a_{\kappa+1}x_{\kappa+1})^{n-\kappa} = 0.$$

For assigned values of $x_1, x_2, \cdots, x_\kappa$, $n-\kappa$ values of $x_{\kappa+1}$ are determined, the equation $I_{\kappa, n-\kappa} = 0$ being satisfied by any $\kappa+1$ values of the whole set of n. Such a set of n values with reference to a *general* symmetric form may be called an "involutive set." The object of Dr. Coble's paper is to determine the number of such involutive sets possessed by the general symmetric form and to determine the conditions upon the form in order that this number shall increase until it attains its maximum, ∞^κ.

A geometrical representation of the form as a spread of order $\kappa+1$ and dimension $n-\kappa-1$ in an $S_{n-\kappa}$ is given. The involutive sets then appear as "apolar n-points" of the spread which lie on a norm-curve $N_{n-\kappa}$. The definition of an apolar n-point is a set of n points any $\kappa+1$ of which are apolar to the spread of order $\kappa+1$.

F. N. COLE,
Secretary.

THE WINTER MEETING OF THE CHICAGO SECTION.

THE twenty-fourth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago on Friday and Saturday, January 1–2, 1909, extending through three sessions. The total attendance was fifty, including the following forty-three members of the Society:

Mr. W. H. Bates, Professor G. A. Bliss, Dr. G. D. Birkhoff, Mr. H. E. Buchanan, Mr. Thomas Buck, Professor W. H. Butts, Professor H. E. Cobb, Dr. A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. E. L. Dodd, Mr. Arnold Dresden, Professor A. B. Frizell, Professor J. W. Glover, Professor A. G. Hall, Mr. T. H. Hildebrandt, Professor T. F. Holgate, Professor O. D. Kellogg, Dr. A. C. Lunn, Dr. E. B. Lytle, Mr. H. F. MacNeish, Dr. W. D. MacMillan, Professor J. L. Markley, Mr. E. J. Miles, Professor G. A. Miller, Professor E. H. Moore, Dr. R. L. Moore, Professor F. R. Moulton, Dr. L. I. Neikirk, Professor Alexander Pell, Mrs. Anna Pell, Professor C. A. Proctor, Mr. A. R. Schweitzer, Miss Ida M. Schottenfels, Professor G. H. Scott, Professor J. B. Shaw, Mr. C. G. Simpson, Professor H. E. Slaught, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor J. W. Young, Professor J. W. A. Young.

The chairman of the Section, Professor G. A. Miller, presided at the afternoon sessions on Friday and Saturday, and Professor E. B. Van Vleck, Vice-President of the Society, at the session on Saturday morning. The session on Saturday afternoon extended to a late hour in order to complete the reading of the papers. The Section has now found it necessary to devote two full days to its regular meetings, and the dinner on the evening of the first day has become an important feature, not only for the promotion of acquaintance, but also as affording an opportunity for the informal discussion of topics indirectly related to the interests of mathematicians.

On this occasion the subject was "The methods of appointment of university professors in foreign countries." The discussion was introduced by Professor Wilczynski, and participated in by Professors Bliss, Curtiss, Dickson, Hall, Holgate, Glover, Miller, Moore, and Van Vleck. This question was again taken up for formal consideration at the close of the session on Saturday morning, and the following resolution was presented by Professor Wilczynski and adopted: Resolved that the Chairman of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY appoint a committee of five to investigate the possibility of improving the character of the mathematical appointments in our colleges and universities, and to make recommendations to the Section at its next meeting. The Chairman appointed the following members of this committee: Professors E. J. Wilczynski, L. E. Dickson, T. F. Holgate, A. G. Hall, and E. B. Van Vleck.

The following officers of the Section were elected for the ensuing year: Professor G. A. Miller, Chairman, Professor H. E. Slaught, Secretary, and Professor O. D. Kellogg, third member of the programme committee.

It was announced that a limited number of the reprints from *Science* of the papers read last year at the joint meetings of mathematicians and engineers are available for distribution among the members of the Society, and will be supplied in the order of application while they last.

The following papers were read at this meeting:

1. Professor E. J. WILCZYNSKI: "Projective differential geometry, fifth memoir."
2. Mr. E. J. MILES: "Determination of the constants in Euler's problem concerning the minimum area between a curve and its evolute."
3. Mr. E. J. MILES: "Surfaces of revolution of minimum resistance."
4. Professor J. W. GLOVER: "A method of analysis of population and vital statistics."
5. Professor G. A. BLISS: "Some exceptional cases in implicit function theory."
6. Professor L. E. DICKSON: "General theory of modular invariants."
7. Professor A. B. FRIZELL: "Common complex numbers as an application of the theory of abstract groups."
8. Dr. A. C. LUNN: "The figures of equilibrium of rotating fluids."

9. Dr. A. C. LUNN: "Some notes on vector analysis."

10. Dr. A. C. LUNN: "The apparent size of a closed curve."

11. Mr. H. E. BUCHANAN: "On certain determinants connected with a problem in celestial mechanics."

12. Mr. H. E. BUCHANAN: "Periodic orbits of three finite bodies" (preliminary report).

13. Professor J. B. SHAW: "Qualitative algebra, second paper."

14. Mr. W. H. BATES: "The Kronecker invariant K of R_λ contained in R_n."

15. Dr. A. L. UNDERHILL: "Note on the calculus of variations."

16. Mr. A. R. SCHWEITZER: "The quaternion as an operator in Grassmann's extensive algebra, second paper."

17. Mr. W. W. DENTON: "On the osculating quartic of a plane curve."

18. Professor E. H. MOORE: "Note on a form of general analysis."

19. Professor J. W. YOUNG: "The notion of a general point field" (preliminary communication).

20. Professor L. E. DICKSON: "On the representation of numbers by modular forms."

21. Professor A. B. FRIZELL: "On the permutations of a series belonging to the second ordinal class."

Mr. Denton's paper was communicated to the Society and read by Professor Wilczyński. Professor Dickson's second paper was read by title. Mr. Buchanan's first paper appeared in full in the February BULLETIN. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the above list.

1. In this paper Professor Wilczynski takes up again the subject of the osculating conics of the plane sections of a surface, deducing some further consequences which were not mentioned in his first paper on this subject at the summer meeting of the Society. He also discusses the Steiner surface which has fourth order contact with a given surface at an arbitrary point. He wishes to emphasize a fact which he then ignored, viz., that one of the principal results of this paper, which is of fundamental importance for projective differential geometry, seems to have been discovered at least twice before and then again lost to the knowledge of mathematicians in general. The theorem

analogous to Meusnier's, which states that the osculating conics of those plane sections of a surface whose planes have a fixed line element in common with the surface form a quadric, was in fact discovered by Moutard in 1863. In 1880 Darboux independently obtained the same theorem, which had in the meantime been forgotten. But this theorem does not seem to be mentioned in any of the larger treatises on the theory of surfaces, nor in the Encyklopädie. The proof given by Professor Wilczynski closely resembles that of Darboux, but is of greater significance owing to the fact that his coordinates have a known geometrical meaning while those of Darboux are merely defined analytically.

2. Given two directed lines and a point A on one of them, the region is determined in which a second point B of the other line must lie, in order that the two points A and B may be joined by a cycloid tangent to the given lines at A and B. Mr. Miles in his first paper considers the following cases: (a) when no cusp of the cycloid is included between the points, (b) when one cusp is included, (c) when n cusps are included. The paper will be offered for publication in the *Annals of Mathematics*.

3. In the second paper of Mr. Miles the surfaces of revolution of minimum resistance, resulting from the resistance laws of Lössl and Duchemin are obtained, and their similarity to the newtonian surface noted. A general law of resistance is then stated which gives surfaces of the same general characteristics as the preceding.

4. Professor Glover develops a method of analysis of population and vital statistics, to determine the effect of tuberculosis and other diseases on the mortality rate, and to estimate the present value of the corresponding monetary loss. Application is made with special reference to the city of Chicago.

5. The nature of the solutions $u = u(x, y)$, $v = v(x, y)$ of a pair of equations

$$x = \phi(u, v), \quad y = \psi(u, v),$$

in the neighborhood of a point where the functional determinant $\Delta = \partial(u, v)/\partial(x, y)$ does not vanish, is well known. If the curve $\Delta = 0$ in the u, v-plane has no singular points, the singular points of the solutions can be classified in a few simple types. In the paper of Professor Bliss this classification is made, and the character of the functions $u(x, y)$ and $v(x, y)$ is discussed.

6. In this paper on modular invariants of one or more forms of any degrees, Professor Dickson avoids the complication which results from the application of linear transformations, or the equivalent annihilators, to functions of the coefficients in constructing invariants. Transformations are employed only to furnish a complete set of non-equivalent classes of the ground forms. Then a polynomial is an invariant if and only if it takes the same value for all the forms in a class. It is shown that the number of linearly independent absolute invariants equals the number of classes under the total group G; that the number of linearly independent invariants, including both absolute and relative, equals the number of classes under the group of transformations of determinant unity. Questions of linear independence are more fundamental in the theory of modular invariants than questions of complete independence. The general theory is applied to the determination of all the invariants of the general m-ary quadratic form in the $GF[p^n]$, there being $2m - 2 + p^n$ linearly independent invariants if $p > 2$; also to the invariants of the binary cubic, which has numerous modular invariants other than powers of its discriminant. For the practical construction of invariants, the general theory is supplemented by a uniform process, of function-theoretic nature, for the conversion of non-invariantive characterizations of the classes into formal invariants. It seems probable that modular invariants are destined to play a rôle in the theory of numbers commensurate with that played by algebraic and differential invariants in other branches of mathematics. The paper has been offered for publication in the *Transactions*.

7. Theoretical arithmetic considers two fundamental rules of combination, one of which is distributive with respect to the other. The natural numbers may be described as a set of symbols which form an abelian semigroup with respect to each rule, the first semigroup possessing a modulus, and which are well ordered by the second rule. The absolute numbers are a set of symbols which form a group with respect to the first rule, a semigroup with respect to the second, and a simply ordered

class with respect to both rules. The real numbers form a group with respect to the second rule; the real numbers excluding zero form a group with respect to the first rule; the set of all real numbers forms a simply ordered class with regard to the second rule.

In this paper Professor Frizell proves the following proposition: Given two rules of combination, of which one is distributive relatively to the other, and given a set of symbols which form a group and a simply ordered class with respect to the second rule, while if we omit the modulus of this group they form a group with respect to the first rule, then it is possible, by the introduction of a single new symbol i, to lay down two new rules, of which the first shall be distributive with respect to the second and which, when applied to the given symbols α, shall coincide respectively with the given rules, in such a way that the whole set of combinations $\alpha \cdot i$ and $(\alpha, \beta \cdot i)$ shall form a group with respect to the second rule and, if we omit its modulus, a group with respect to the first rule. The new set of compound symbols, however, is no longer simply ordered according to either rule.

8. Dr. Lunn's first paper gives a theorem related to those of Hamy and Volterra, showing that a mass of rotating fluid could not be in equilibrium, with its mass so distributed that the internal equipotential surfaces are confocal ellipsoids, even if the angular velocity varies with the distance from the axis.

9. Dr. Lunn's second paper gives: (*a*) a direct deduction of Rodrigues's parameter representation of an orthogonal matrix by integration of the vector differential equation defining the distribution of velocity in a rigid body; (*b*) a proof that a solenoidal vector can in general be represented as the vector product of two potential vectors; (*c*) some formulas on the vector potential analogous to those for the scalar potential, especially relating to functions of points and lines analogous to Green's function.

10. Dr. Lunn's third paper gives a formula for the solid angle subtended at a given point by a closed curve in space, as the scalar line integral around the curve of a suitably determined vector point function. The corrections for certain possible discontinuities are discussed and a number of examples worked out.

12. In his second paper Mr. Buchanan considers periodic orbits of three finite bodies near the lagrangean straight line orbits. Certain convenient parameters are introduced, the characteristic exponents are found and the linear terms are discussed. It is expected in a future report to give a full discussion of the problem.

13. Professor Shaw's paper discusses general multiplication. The subdivisions of the subject are (1) Lineal multiplication, which is independent of the unit system. There are three classes of laws: (*a*) those depending on limitation types;[*] (*b*) those due to invariant expressions in symmetric groups, expressions corresponding to the single units in the quadrates of order 1 in the Frobenius algebra corresponding to the group;[†] (*c*) each limitation type determines a subgroup of $G_{n!}$, and each subgroup has symmetric expressions, corresponding to the single units in the quadrates of order 1 in its Frobenius algebra. Examples are Grassmann's and Gibbs's systems. (2) Orthogonal multiplication, which is not altered by orthogonal changes of the units. Examples are McAulay's multernions.[‡] (3) Linear homogeneous multiplication, which is not altered by linear homogeneous transformations of the units. Examples are given. (4) Generalized multiplication, which is independent of the units. The paper will be offered for publication in the *Transactions*.

14. Professor Maschke, in the *Transactions*, volume 6, pages 87–93, calculated, for even values of λ, the Kronecker invariant K of a space of λ dimensions R_λ, represented as a differential parameter of a space of higher dimensions R_n containing R_λ. At the close of this calculation, he says that the principles which lead to this expression of K for even values of λ will doubtless also be sufficient to solve the more complicated problem for the case of odd values of λ. In the present paper, Mr. Bates calculates the expression of K for odd values of λ.

15. The solution of the Jacobi equation, in connection with the minimizing of

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[*] Shaw, Synopsis of Linear Associative Algebra, p. 77.

[†] Ibid., pp. 49–51.

[‡] *Proc. Royal Soc. of Edinburgh*, 1907–8, vol. 28, pp. 503–585. Joly: *Proc. Royal Irish Acad.*, 1897 (3) 5, pp. 73–123.

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt,$$

may present difficulties. In volume 9, page 336, of the *Transactions*, and in a note presented to the Chicago Section, December, 1907, methods were given wherewith one may avoid the solution of the Jacobi equation, and yet obtain information as to the extent of the interval in which a weak minimum exists. In the present note Dr. Underhill applies these methods to several of the standard problems.

16. Mr. Schweitzer defines the vectors

$$\begin{aligned} X &= x_1E_1 + x_2E_2 + x_3E_3 + x_4E_4, \\ i_3X &= \lambda_xE_1 + \mu_xE_2 + \nu_xE_3 + \omega_xE_4, \\ j_3X &= \lambda'_xE_1 + \mu'_xE_2 + \nu'_xE_3 + \omega'_xE_4, \\ k_3X &= \lambda''_xE_1 + \mu''_xE_2 + \nu''_xE_3 + \omega''_xE_4, \end{aligned}$$

where

$$\begin{aligned} \xi_x &= \xi_1x_1 + \xi_2x_2 + \xi_3x_3 + \xi_4x_4, \\ \xi &= \lambda, \lambda', \lambda''; \ \mu, \mu', \mu''; \ \nu, \nu', \nu''; \ \omega, \omega' \ \omega''; \end{aligned}$$

and expresses

$$\begin{aligned} i_3X &= i_2X + j'_1X + k''_1X, \\ j_3X &= j_2 + k'_1X + i''_1X, \\ k_3X &= k_2X + i'_1X + j''_1X \end{aligned}$$

by suitable definitions, and then determines the operators i_3, j_3, k_3 in the form

$$\begin{aligned} i_3 &= \lambda_2 i_{\epsilon_2} + \lambda_1 + \lambda_3 j_{\epsilon_{11}} + \lambda_4 k_{\epsilon_{12}}, \\ j_3 &= \lambda_3 j_{\epsilon_2} + \lambda'_1 + \lambda'_4 k_{\epsilon_{11}} + \lambda'_2 i_{\epsilon_{12}}, \\ k_3 &= \lambda''_4 k_{\epsilon_2} + \lambda''_1 + \lambda''_2 i_{\epsilon_{11}} + \lambda''_3 j_{\epsilon_{12}}, \end{aligned}$$

where

$$\epsilon_2^2 = \epsilon_{11}^2 = \epsilon_{12}^2 = 1,$$

by the specification that the outer products $[X \cdot i_2X \cdot j_2X \cdot k_2X]$, $[X \cdot j'_1X \cdot k'_1X \cdot i'_1X]$, $[X \cdot k''_1X \cdot i''_1X \cdot j''_1X]$ shall be identically equal respectively to

$$\rho_2(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4,$$

$$\rho'_1(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4,$$

$$\rho''_1(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2E_1E_2E_3E_4,$$

$\rho_2, \rho_1', \rho_1''$ being constants. The operators $i_{\epsilon_2}, j_{\epsilon_2}, k_{\epsilon_2}$; $i_{\epsilon_{11}}, j_{\epsilon_{11}}, k_{\epsilon_{11}}$; $i_{\epsilon_{12}}, j_{\epsilon_{12}}, k_{\epsilon_{12}}$ constitute (with unity) three systems of hamiltonian units. The systems have the "same sense" if $\epsilon_2 = \epsilon_{11} = \epsilon_{12}$ and all the systems are "right-handed" if $\epsilon_2 = \epsilon_{11} = \epsilon_{12} = -1$. Thus the function $\rho(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$ may be said to be *characteristic* of the quaternion as an operator in Grassmann's extensive algebra.

(17) The projective differential geometry of a plane curve which is not a straight line is equivalent to the theory of the invariants and covariants of the linear homogeneous differential equation of the third order

$$\frac{d^3y}{dx^3} + 3p_1\frac{d^2y}{dx^2} + 3p_2\frac{dy}{dx} + p_3y = 0.$$

Mr. Denton has computed the equation of the osculating quartic, *i. e.*, the quartic having contact of the highest possible order with the given curve in a given non-singular point. An invariant triangle which was first introduced by Professor Wilczynski is used as triangle of reference. The calculation, which was long and laborious, was carried out completely by two independent methods, thus furnishing an adequate check for the final explicit equation. The paper will be offered for publication in the *Transactions.*

(18) The note of Professor Moore, to be published in the BULLETIN, gives a brief synopsis of his lectures, "On the theory of bilinear functional operations," delivered at the New Haven Colloquium, September, 1906, together with an account of subsequent developments and applications of the central thought of those lectures.

(19) Professor Young's paper describes what he calls a general point field, which is the geometric counterpart of the notion of a general number field. A net of rationality * and a chain in n-dimensions are special types of point fields of n-dimensions. A point field on a line is a class of points isomorphic with a number field; i. e., if three points of the field are labelled 0, 1, ∞ respectively, and the corresponding number system developed,† the numbers corresponding to the points of the given field form a field (of numbers). The synthetic definition of such fields is discussed. Fields of n-dimen-

sions are defined in terms of linear fields, and applications to various analytic problems are indicated. A general theory of linear fields defined by three points is discussed under general hypotheses.

(20) The second paper by Professor Dickson proves for the cases $m \leqq 3$ that every form of degree m in $m+1$ variables vanishes for values, not all zero, in any finite field. The larger part of the paper relates to forms in two or more variables which represent exclusively cubes in a finite field. The paper will appear in the BULLETIN.

(21) If a series of type ω be re-arranged by bringing each element in succession to the first place and then treating the remainder of each new series in the same way, the resulting permutations form a series of type $\omega_1 = \omega^\omega$. Now the set of all conceivable arrangements of the series of natural numbers has the cardinal of the continuum, but every possible arrangement of the natural numbers forms a series belonging to the second class of ordinal numbers. Professor Frizell shows, in his second paper, how to exhibit the set of natural numbers as an ordinal type higher than ω_1, and proves that it is not possible to get outside the second ordinal class by permuting the elements of any series of type lower than Ω.

H. E. SLAUGHT,
Secretary of the Section.

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* Cf. Veblen and Young, "A set of assumptions for projective geometry." *Amer. Jour. of Mathematics*, vol. 30 (1908), p. 356ff.

† Veblen and Young, loc. cit., p. 352.

THE SIXTIETH MEETING OF THE AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE.

THE sixtieth meeting of the American Association for the Advancement of Science was held at Johns Hopkins University during the convocation week, December 28, 1908, to January 2, 1909. The president of the meeting was Professor T. C. Chamberlin, University of Chicago. The address of the retiring president, Professor E. L. Nichols, entitled "Science and the practical problems of the future," was given at the Peabody Institute on the evening of the opening day.

Comparatively few papers on pure mathematics appeared on the program of Section A because of the fact that the AMERICAN MATHEMATICAL SOCIETY held its annual meeting in affiliation with the Association. The address of the retiring vice-president, President E. O. Lovett, Rice Institute, Houston, Texas, was read by the secretary of the section. It was entitled "The problem of several bodies, recent progress in its solution," and an abstract of it has appeared in a recent number of *Science* (January 15, 1909).

The officers of the section were: vice-president, C. J. Keyser; secretary, G. A. Miller; councilor, G. B. Halsted; member of the general committee, F. R. Moulton; sectional committee, E. O. Lovett, C. J. Keyser, G. A. Miller, E. B. Frost, Harris Hancock, F. R. Moulton, E. W. Brown. On the recommendation of the sectional committee the following thirty-six members of the AMERICAN MATHEMATICAL SOCIETY were elected fellows of the Association: Joseph Allen, R. B. Allen, Oskar Bolza, W. H. Bussey, B. E. Carter, Abraham Cohen, E. H. Comstock, H. A. Converse, S. A. Corey, F. F. Decker, C. C. Engberg, F. C. Ferry, William Gillespie, C. O. Gunther, U. S. Hanna, A. E. Haynes, W. J. Hussey, Kurt Laves, A. H. McDougall, Max Mason, W. F. Osgood, J. M. Page, M. T. Peed, James Pierpont, W. J. Rush, P. L. Saurel, G. T. Sellew, E. B. Skinner, D. E. Smith, P. F. Smith, R. P. Stephens, H. D. Thompson, E. B. Van Vleck, Oswald Veblen, H. S. White, F. S. Woods.

In addition to the address of the vice-president, the following sixteen papers were read before the section:

1. Mr. V. M. SLIPHER: "The spectrum of Mars."
2. Professor E. B. FROST and Mr. J. A. PARKHURST: "Spectrum of comet Morehouse."
3. Professor E. E. BARNARD: "On the changes in the tail of comet Morehouse."
4. Professor FRANK SCHLESINGER: "The orbit of the Algol type variable Delta Libræ."
5. Professor MILTON UPDEGRAFF: "The 6-inch transit circle of the U. S. Naval Observatory."
6. Professor F. R. MOULTON: "On certain implications of possible changes in the form and dimensions of the sun, and some suggestions for explaining certain phenomena of variable stars."
7. Mr. R. H. BAKER: "On the spectra of Alpha Virginis and similar stars."
8. Mr. F. C. JORDAN: "The orbit of Alpha Coronæ Borealis."
9. Professor E. B. FROST: "Radial velocities in Professor Boss's star stream in Taurus."
10. Messrs. PHILIP FOX and GEORGIO ABETTI: "The interaction of sun spots."
11. Professor HARRIS HANCOCK: "Elliptic realms of rationality."
12. Dr. ARTEMAS MARTIN: "Algebraic solution of the 'three point' problem."
13. Professor J. B. WEBB: "Esperanto and a sexdecimal notation."
14. Professor J. A. MILLER and Mr. W. R. MARRIOTT: "Comet Morehouse."
15. Dr. L. A. BAUER: "On the interpolation formula of geophysics."
16. Mr. H. W. FISK: "A graphical aid to the determination of latitude and azimuth from Polaris observation."

In the absence of their authors the paper by Mr. Slipher was read by J. A. Miller; the joint paper by E. B. Frost and J. A. Parkhurst was presented by W. S. Eichelberger; the three papers by Frank Schlesinger, E. H. Baker and F. C. Jordan, respectively, were read by J. A. Brashear; F. R. Moulton's paper was read by E. D. Roe; G. F. Hull presented the two papers by E. B. Frost and by Philip Fox and Georgio Abetti, the papers by E. E. Barnard and Artemas Martin were read by title. The remaining papers were read by their authors. The following abstracts of those which were of mathematical interest bear numbers corresponding to those of the titles in the list. The abstracts of the others appeared in the report of Section A of this meeting published in *Science*, January 22, 1909.

6. The problems treated in the paper by Professor Moulton are: (1) The theoretical shape of the sun, (2) the character and period of its possible gravitational oscillations, (3) the effects of changes of its dimensions upon its rate of rotation, (4) its energy of rotation, (5) its potential energy, (6) its temperature and rate of rotation, and (7) applications of the same ideas to variable stars.

The results are: (1) The sun is oblate and the theoretical difference in its polar and equatorial diameters is less than 0''.01. (2) Its gravitational oscillations are expressible in spherical harmonics whose periods depend upon their order. Assuming the sun to be a homogeneous liquid, the longest period is 3 h. 8 m. If it has the viscosity of water, this oscillation will change to 37 per cent. of its value in 2.2×10^{15} years. (3) The change of the sun's diameter by 0''.1 will change its period of rotation by 7.8 minutes. (4) The formula was found for the change in the rotational energy. (5) The formula for the potential of a spheroid of polar radius c, equatorial $c\sqrt{1+\lambda^2}$, and mass m upon itself is

$$V \doteq \tfrac{3}{5}k^2\frac{m^2}{c}\left(1 - \tfrac{1}{3}\lambda^2 - \tfrac{7}{15}\lambda^4 - \cdots\right).$$

(6) The expansion of the sun by 0''.1 will decrease its temperature (assuming its specific heat is unity) more than 1400° C., and if it obeys Stefan's law, diminish its radiation (assuming its temperature to be 6000° before expansion) by more than 65 per cent. (7) It is shown how gravitational oscillations can explain many puzzling phenomena of variable stars, such as variable periods in the so-called eclipse variables, secondary maxima and minima, varying maxima and minima, etc. It is thought that these factors are supplementary even in those cases where the binary character of the star is certain, and that perhaps in certain classes of stars they may be the only causes of variability.

11. It is known that on the Riemann surface associated with

$$s = \pm\sqrt{A(z - a_1)(z - a_2)(z - a_3)(z - a_4)},$$

in which a_1 and a_2 are connected by a canal as are also a_3 and a_4, every one-valued function of position which has everywhere a definite value is of the form $w = p + q\ s$, where p and q are rational functions of the complex variable z; and reciprocally every function of this form is a one-valued function of position on this Riemann surface. If we denote two such functions by

$$w_1 = p_1 + q_1 \cdot s, \quad w_2 = p_2 + q_2\ s,$$

then the sum, difference, product and quotient of the two functions w_1 and w_2 are functions of the form $w = p + q \cdot s$.

Let z take all complex values and consider the collectivity of all rational functions of z with arbitrary constant real or complex coefficients. These functions form a closed realm, the individual functions of which repeat themselves through the processes of addition, subtraction, multiplication and division, since clearly the sum, the difference, the product and the quotient of two or more rational functions is a rational function and consequently an individual of the realm. This realm is denoted by (z).

It is evident that if we add (or *adjoin*) the algebraic quantity s to this realm we will have another realm, the individual functions or elements of which repeat themselves through the processes of addition, subtraction, multiplication and division. This realm includes the former realm. We shall call it the elliptic realm and denote it by (s, z).

By a theorem due to Liouville, the most general one-valued doubly periodic function is a rational function of z and s. It is consequently a one-valued function of position on the Riemann surface and belongs to the elliptic realm of rationality (z, s).

The elliptic or doubly periodic realm of rationality (z, s) degenerates into the simply periodic realm when any pair of branch points are equal, say $a_1 = a_2$; and into the realm of rational functions when two pairs of branch points are equal, say $a_1 = a_2$ and $a_3 = a_4$.

Thus the elliptic realm includes the three classes of one-valued functions: (1) the rational functions, (2) the simply periodic functions, (3) the doubly periodic functions. All these one-valued functions, and only these, have algebraic addition theorems.

In other words, *all functions of the realm* (z, s) *have algebraic addition theorems, and no one-valued function that does not belong to this realm has an algebraic addition theorem.*

We have thus the theorem:
The one-valued functions of position on the Riemann surface

$$s^2 = A(z - a_1)(z - a_2)(z - a_3)(z - a_4)$$

belong to the closed realm (z, s) *and all elements of this realm and no others have algebraic addition theorems.*

Professor Hancock's paper will be offered to the *American Journal of Mathematics* for publication.

12. The paper by Artemas Martin is devoted to an algebraic determination of the point within a triangle at which the sides subtend given angles. The paper is to appear in the *Mathematical Magazine*, edited by the author.

13. The paper by J. Burkitt Webb is devoted to exhibiting the advantages which would result from the adoption of a system of notation with 16 as its base. The success which has attended the movements towards a universal language has inspired the author with hope in the success of a movement towards the selection of a more useful system of notation, and he pointed out the many advantages which the base 16 would offer.

15. The rather prevalent custom of resolving or expressing every natural phenomenon — be it periodic or otherwise — by a Bessel or a Fourier series, or by spherical harmonic functions, has brought about at times, especially in geophysical and cosmical phenomena, if not direct misapplications, at least misinterpretations of the meaning and value of the derived coefficients. Instead of clarifying the situation our calculations may have actually contributed to befog it. Instead of rejecting, one must learn to consider the outstanding residuals as the *true* facts of nature and not treat them as though they were "abnormal" or contrary to nature's law.

Dr. Bauer exemplified these statements in a brief discussion of two cases that are typical in geophysical investigations — the one involving an application of spherical harmonic functions to the representation of the distribution of the earth's magnetism over the earth, while the other involved the use of Fourier series in the representation of certain diurnal geophysical phenomena.

The chief purpose of the paper was to call renewed attention to the limitations, from a physical standpoint, of the form of "interpolation formulas" usually employed in the representation of natural phenomena. The formula of Gauss is simply a mathematical approximation to the actual law of distribution. In spite of all the work done, no one up to date has succeeded in giving any physical interpretation of the coefficients of the harmonic terms beyond those of the first order involving three coefficients, and these three stand for the simplest possible case of a first approximation to the actual state of the earth's magnetism: that of a uniform magnetization about a diameter inclined to the axis of rotation.

16. Mr. H. W. Fisk considers the formula for latitude,

$$\phi = h - p \cos t + \tfrac{1}{2}p^2 \sin 1' \sin^2 t \tan h + \cdots$$

from Chauvenet's Astronomy I., § 176, and the formula for azimuth,

$$A = p \sin t \sec \phi + \tfrac{1}{2}p^2 \sin 1' \sec \phi \tan \phi \sin 2t + \cdots$$

from Jordan, "Zeit und Orts-Bestimmung," page 122. The first terms of these formulas are readily computed. The last terms, called correction terms, are arranged as a set of curves from which the value is quickly taken by inspection. The geographical limits within which this method may be used, as well as the expected accuracy under different conditions are discussed. Attention is given to the change in correction terms due to the progressive change in the value of p. Assuming, for instance, that p will continue indefinitely to change at its present rate 0′.34 per annum, we may find the period during which the curves could be used within the error 0′.1. The latitude curves are found to be good 83 years at latitude 10° and about 8 years at latitude 60°. The azimuth curves are correspondingly useful for 80 years at 10° and 4 years at 60°.

The next regular meeting of the association will be held in Boston under the presidency of President D. S. Jordan, Stanford University. No summer meeting will be held during 1909, in order to permit the members to attend the meetings of the British Association at Winnipeg, which has extended a cordial official invitation to the members of the American Association. Professor E. W. Brown, Yale University, was elected vice-president and chairman of Section A, and Professor G. A. Miller, University of Illinois, continues as secretary. The section elected Professor Winslow Upton, Ladd Observatory, as member of the sectional committee for five years.

G. A. MILLER,
Secretary.

SOME SURFACES HAVING A FAMILY OF HELICES AS ONE SET OF LINES OF CURVATURE.*

BY MISS EVA M. SMITH.

IN a recent paper, Forsyth † gives a general method for the determination of surfaces with assigned lines of curvature, and he solves completely the case where both sets are circles. We apply the method to the case where one of the given sets consists of helices, and it appears that surfaces do exist having as one set of lines of curvature general helices ($\rho/\tau =$ constant along each curve), but there are always limitations on the forms of ρ and τ. In particular, ρ and τ cannot both be constant along every curve. The complete solution seems to be too wide for analytic discussion, but there are two particular cases for which definite results can be obtained. This note contains a discussion of these cases.

1) Assuming that ρ and τ are constant along each curve of the set (regular helices), we obtain the result that: *There are no surfaces with regular helices as one set of lines of curvature.*

2) If ρ/τ is constant along each curve of one set of lines of curvature, and the other set consists of geodesics, we can obtain a complete solution; the equations of the resulting surfaces in parametric form are given at the end of this paper. The notation and equations used are those given in Darboux, Théorie des surfaces, volume 2, but derivatives with respect to u and v are here denoted by suffixes 1 and 2 respectively.‡

§ 1.

Consider the case where the helices are all regular. The curves $v =$ constant are helices, and therefore ρ and τ for these curves are functions of v only, and we denote τ/ρ by k.

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* For the suggestion of this subject I am indebted to Prof. A. R. Forsyth.

† *Messenger of Mathematics*, vol. 38 (1908), pp. 33–44.

‡ Note that p_1 is not $\partial p/\partial u$. To express derivatives of the rotations we use parentheses, *e. g.* $(p)_1 = \partial p/\partial u$.

We have the following set of equations*:

$$p = 0, \quad q_1 = 0,$$

$$(1) \quad \frac{1}{\tau} = \frac{1}{A}\phi_1,$$

$$(2) \quad r = \frac{\sin\phi}{\rho}A, \qquad (2') \quad r = -\frac{1}{C}A_2,$$

$$(3) \quad q = -\frac{\cos\phi}{\rho}A, \qquad (4) \quad p_1 = \frac{1}{r}(q)_2,$$

$$(5) \quad r_1 = -\frac{1}{q}(p_1)_1, \qquad (5') \quad r_1 = \frac{1}{A}C_1,$$

where ϕ is the angle between the osculating plane of the curve $v =$ constant and the normal plane to the surface. From (2), (3), (4), (5) we obtain

$$(7) \quad r_1 = 2\frac{\phi_{12}}{k\phi_1}\sec\phi + \frac{k_2}{k^2}\operatorname{cosec}^2\phi\sec\phi - \frac{1}{k\phi_1}\left(\frac{\phi_{12}\operatorname{cosec}\phi}{\phi_1}\right)_1$$

Also from (1), (2′), (5′),

$$(8) \qquad r_1 = \frac{1}{k}\left[\frac{\tau_2}{\tau}\cot\phi\operatorname{cosec}\phi - \frac{1}{\phi_1}\left(\frac{\phi_{12}\operatorname{cosec}\phi}{\phi_1}\right)_1\right].$$

From (7) and (8)

$$\frac{\phi_{12}}{\phi_1} = \tfrac{1}{2}\left(\frac{\tau_2}{\tau}\cot^2\phi - \frac{k_2}{k}\operatorname{cosec}^2\phi\right).$$

Differentiating with respect to u and substituting in (8), we get

$$(9) \qquad r_1 = \tfrac{3}{2}\cot\phi\operatorname{cosec}\phi\left[\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) + \frac{\tau_2}{\tau}\right].$$

Combining (9) with (6) we obtain, after some reduction,

$$6\operatorname{cosec}^4\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) - \tfrac{3}{2}\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{3k_2}{k}\right)$$
$$+ \frac{k^3}{2}\left[\operatorname{cosec}^2\phi\left(\frac{\tau_2}{\tau} - \frac{k_2}{k}\right) - \frac{\tau_2}{\tau}\right] = 0.$$

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* Darboux, l. c., pages 383, 386.

This equation determines ϕ as a function of v only, but as the curves $v =$ constant are not plane curves, ϕ is a function of both u and v, hence the equation must reduce to an identity. Equating to zero the coefficients of the various powers of cosec ϕ, we get

$$\frac{\tau_2}{\tau} = 0, \quad \frac{\tau_2}{\tau} - \frac{k_2}{k} = 0,$$

or $\rho =$ constant, $\tau =$ constant. Substituting in the preceding equations, we obtain finally $C = 0$. Hence no surfaces exist of the kind we seek.

§ 2

Let us now assume that $\rho/\tau = \alpha$, a function of v only, and that the curves $u =$ constant are geodesics. Having regard to the properties of geodesics, we may put $C = 1$.

The equations referred to above now become

$$p = 0, \quad q_1 = 0, \quad r_1 = 0,$$

$$\text{(i)} \quad r = \frac{\sin\phi}{\rho} A, \qquad \text{(i}'\text{)} \quad r = -A_2,$$

$$\text{(ii)} \quad q = -\frac{\cos\phi}{\rho} A, \qquad \text{(iii)} \quad (r)_2 + qp_1 = 0,$$

$$\text{(iv)} \quad p_1 = \frac{1}{r}(q)_2, \qquad \text{(iv}'\text{)} \quad p_1 = V.^*$$

There are two cases to be considered according as V is zero or not. If V is not zero, take a new variable t so that $Vdv = dt$. From (iii), (iv)

$$\frac{\partial^2 r}{\partial t^2} + r = 0, \quad r^2 + q^2 = U'^2,$$

$$\therefore r = U' \sin(t + U); \quad q = -U' \cos(t + U).$$

From (i), (ii)

$$t + U = \phi, \quad -A_2 = U' \sin(t + U).$$

We can absorb U' into A, which is equivalent to reducing U' to unity, and we obtain finally

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* Capital letters U, V denote functions of the variables u alone, v alone respectively.

$$A = \rho, \quad C = 1, \quad \rho_2 = -\sin(t + U), \quad U = \alpha u.$$

The last result shows that α, which is a function of v only, must be an absolute constant; also ρ must be of the form $a \cos \alpha u + b \sin \alpha u$, where a and b are functions of v alone and

$$a_2 = -\sin t, \quad b_2 = -\cos t.$$

All the conditions are now satisfied, so that surfaces do exist having the required properties, and to obtain them we have the equations

$$x_1^2 + y_1^2 + z_1^2 = \rho^2, \quad x_1 x_2 + y_1 y_2 + z_1 z_2 = 0, \quad x_2^2 + y_2^2 + z_2^2 = 1.$$
$$D = \rho^2 \cos(t + ku), \quad D' = 0, \quad D'' = \rho V^*$$

These can be integrated and we obtain as coordinates of a point on a surface referred to a special set of axes

$$x = \rho_1 \frac{\sin(\sqrt{1 + \alpha^2}.u)}{\sqrt{1 + \alpha^2}} - \rho \cos \sqrt{(1 + \alpha^2}\ u),$$

$$y = \rho_1 \frac{\cos(\sqrt{1 + \alpha_1^2}.u)}{\sqrt{1 + \alpha^2}} + \rho \sin(\sqrt{1 + \alpha^2}.u),$$

$$z = \frac{\rho_1}{\alpha\sqrt{1 + \alpha^2}}.$$

From a further consideration of the equation given on pages 383, 386, l. c., it appears that the curves $u =$ constant are plane curves, whose radii of curvature are given by

$$1/\rho = p_1 = V.$$

This shows that the set $u =$ constant consists of the different positions of an invariable curve in a plane which moves in a direction perpendicular to itself.

If $V = 0$, we see at once that the surface is a developable whose coordinates may be written in the form

$$x = a + lv, \quad y = b + mv, \quad z = c + nv,$$

where l, m, n, a, b, c are all functions of u only, and in virtue

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[*] Darboux, l. c., page 378.

of the special properties of the curves $u=$ constant, $v=$ constant, satisfy the following relations:

$$l^2+m^2+n^2=1, \quad l_1^2+m_1^2+n_1^2=1,$$

$$\frac{a_1}{l_1}=\frac{b_1}{m_1}=\frac{c_1}{n_1}=\beta, \text{ say.}$$

Then we have $A=v+\beta$ and $ds^2=(v+\beta)^2\,du^2+dv^2$. The equations to determine p, q, r, p_1, q_1, r_1, now become

$$p=0, \quad q_1=0, \quad r_1=0, \quad p_1=0,$$

$$(1) \quad r=-1, \qquad r=\frac{\sin\phi}{\rho}(v+\beta),$$

$$(2) \quad q=U, \qquad (2') \quad q=-\frac{\cos\phi}{\rho}(v+\beta),$$

$$(3) \quad \tau\phi_1=v+\beta.$$

Combining (1), (1′), (2), (2′) we obtain

$$\tan\phi=\frac{1}{U},$$

which, upon differentiation with respect to u, and substitution from (3), leads to the following equation:

$$\alpha=\frac{\rho}{\tau}=\frac{U_1}{(1+U^2)^{\frac{3}{2}}}.$$

From this it is apparent that ρ/τ, which is a function of v only, is an absolute constant. By integration we obtain

$$\frac{U}{\sqrt{1+U^2}}=\alpha u+\gamma \qquad (\gamma=\text{constant}).$$

No generality is lost by making $\gamma=0$, and then

$$U=\frac{\alpha u}{\sqrt{1-\alpha^2u^2}}.$$

The equations to be integrated are

$$\sum l^2 = 1, \quad \sum l_1^2 = 1, \quad \frac{l_1}{a_1} = \frac{m_1}{b_1} = \frac{n_1}{c_1} = \frac{1}{\beta},$$

$$\begin{vmatrix} l & m & n \\ l_1 & m_1 & n_1 \\ l_{11} & m_{11} & n_{11} \end{vmatrix} = U$$

and the surface is defined by the equations

$$x = a + kuv,$$

$$y = b + \frac{kuv}{U\sqrt{1-k^2}}(\cos\psi + k\sin\psi\ U),$$

$$z = c + \frac{kuv}{U\sqrt{1-k^2}}(\sin\psi - kU\cos\psi),$$

where a, b, c are perfectly determinate functions of u, and

$$k = \frac{\alpha}{\sqrt{1+\alpha^2}}, \quad \sin k\psi = \frac{ku}{\sqrt{1-k^3}}.$$

BRYN MAWR COLLEGE, PENNA.,
December, 1908.

NOTE ON ENRIQUES'S REVIEW OF THE FOUNDATIONS OF GEOMETRY.

BY MR. A. R. SCHWEITZER.

(Read before the Chicago Section of the American Mathematical Society, April 18, 1908.)

1. As is well known, a continuous descriptive space may be extended to a projective one by the suitable definition of the projective points. For the definition of euclidean geometry on the basis of descriptive-projective geometry, we distinguish between the proper and improper points of the space and the assumption is made that all the improper points lie on the unique improper plane. This assumption may be provided for by means of a

suitable descriptive axiom. The following question then arises:* Is it possible to define a unique euclidean geometry on the basis of the preceding descriptive-projective space without introducing further indefinables or axioms? In the present note I show that such a definition is possible if we employ the notion of a propositional function.†

2. The conventional manner of defining euclidean geometry on the basis of descriptive-projective geometry is to select an elliptic polar system, say Σ_0, in the unique improper plane and to define perpendicularity, congruence, etc., with reference to it. Every such elliptic polar system Σ_0, however, gives rise to a euclidean geometry $G(\Sigma_0)$. A typical congruence proposition in this geometry will have the form:‡ "If . . . with reference to Σ_0, then . . . with reference to Σ_0." Thus the class of elliptic polar systems $\{\Sigma_0\}$ in the improper plane gives rise to a unique class of euclidean geometries $\{G(\Sigma_0)\}$. The members of this latter class are, however, given by means of the geometry function $G(\Sigma)$ of a variable elliptic polar system Σ in the improper plane. This geometry function is then uniquely defined. A typical congruence proposition of $G(\Sigma)$ has the form: "If . . . with reference to Σ then . . . with reference to Σ." The terms of the latter implication are propositional functions of the variable Σ. As Russell maintains,§ such terms are not propositions but yield propositions for every constant value of the variable; the implication which involves the propositional functions as terms is, however, a proposition and with reference to the implication, in the terminology of Peano, Σ is an apparent variable.

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* Compare Enriques, Encyklopädie der mathematischen Wissenschaften, vol. III, 1, p. 34.

† See B. Russell, The Principles of Mathematics, pp. 13, 82.

‡ The spaces indicated are to be filled out suitably by congruence properties.

§ l. c., p. 13. In this connection, see some interesting points brought out by a controversy between McColl and Russell in the journal *Mind*, n. s. vol. 17 (1908), pp. 151–152; 300–302.

NOTES.

The opening (January) number of volume 31 of the *American Journal of Mathematics* contains the following papers: "The quadric spreads connected with the configuration $\Gamma^{n+2}_{n+4,n}$ and a special case in the Pascal hexagram," by W. B. Carver; "The group membership of singular matrices," by A. Ranum; "Methods to determine the primitive roots of a number," by G. A. Miller; "Standard forms of certain types of Peirce algebras," by J. B. Shaw; "On the law of gravitation in the binary systems," by F. L. Griffin; "Lösung des Lehmer'schen Problems," by E. Landau.

The next meeting of the Deutsche Mathematiker-Vereinigung will be held at Salzburg.

At the meeting of the London mathematical society held on January 14, 1909, the following papers were read: By H. Hilton, "Canonical form of a linear substitution"; by J. Hammond, "On the solutions of the quintic"; by A. Cunningham, "On octavic and sexadecimic residuacity"; by E. W. Hobson, "On change of the variable in a Lebesgue integral"; by F. H. Jackson, "On Abel's extension of Taylor's series"; by H. M. Macdonald, "Note on the evaluation of a certain integral containing Bessel's functions."

At the meeting of the society held on February 11 the following papers were read: By A. C. Dixon, "On the relation between Pfaff's problem and the calculus of variations"; by W. H. Young, "On implicit functions and their differentials"; by W. H. Salmon, "On a certain family of cubic surfaces"; by E. W. Hobson, "Some fundamental properties of Lebesgue integrals of a two dimensional domain"; by L. E. Dickson, "Modular invariants of a general system of linear forms."

The executive committee of the international commission of secondary instruction in mathematics held its first session at Cologne in September, 1908, and perfected its organization. The principal features are the following:

Each country having at least two delegates at two or more congresses of mathematicians will have one vote in the commission; other countries may send one delegate, who will have no vote. The executive committee is final authority on all questions concerning the commission. The official organ will be *L'Enseignement Mathématique.* The purpose of the commission is to institute an inquiry and to prepare a comprehensive report concerning the aims and methods of mathematical instruction throughout the world. The commission hopes to present its complete report to the fifth international congress, to be held at Cambridge, England, in August, 1912.

THE Spanish association for the advancement of science was organized last year; the first meeting was held at Zaragossa, October 22–29, 1908. Sixteen papers were read in the section of mathematics. Two committees were appointed, one to examine and report on the feasibility of organizing a Spanish mathematical society, the other to study the formation of a Spanish mathematical vocabulary.

THE royal academy of sciences of Turin announces the following programme for the seventeenth award of the Bressa prize. The purpose of the competition is to reward the scientist or inventor, irrespective of affiliation or nationality, who, during the period 1907–1910, has in the judgment of the academy made the most brilliant and useful discovery or who has made the most noteworthy advances in physical or experimental sciences, natural history. pure or applied mathematics, chemistry, physiology or pathology, not excluding geology, history, geography, or statistics.

Competing memoirs should be printed and sent to the president with the statement that they are entered for the competition, which closes December 31, 1910. The net value of the prize is 9300 francs.

THE following courses in mathematics will be offered during the summer semester, 1909.

UNIVERSITY ON GÖTTINGEN. — By Professor F. KLEIN: Mechanics of the continuum, four hours; Seminar, two hours. — By Professor D. HILBERT: Theory of numbers, four hours; Selected problems in the theory of functions, two hours; Seminar, two hours. — By Professor C. RUNGE: Differential equations, four hours; Seminar, two hours. — By Professor W. VOIGT: Thermodynamics, four hours. — By Professor L.

PRANDTL: Theory of aeronautics, four hours. — By Professor A. SCHWARZSCHILD: Advanced celestial mechanics, four hours. — By Professor E. WIECHERT: Theory of potential, four hours. — By Professor E. ZERMELO: Differential and integral calculus, four hours. — By Dr. O. TOEPLITZ: Analytic geometry, four hours. — By Dr. P. KOEBE: Elementary mathematics from a higher standpoint, four hours.

DURING the year 1908 the following doctorates were conferred at the University of Paris with mathematics as the major subject (the titles of the theses are given): T. LALESCU (Bucarest), "Sur l'équation de Volterra"; J. B. ROUSIER (Vienna), "Ondes par émersion"; C. POPOVICI (Jasey, Roumania), "Sur les surfaces intégrales communes aux équations différentielles"; N. ENACHE (Dozesci, Roumania), "Contributions à la théorie de l'écoulement sur les déversions à minces parois et à nappe noyée au dessous"; H. B. HEYWOOD (Peklam, England), "Sur l'équation fonctionnelle de Fredholm et quelques-unes de ses applications."

AT the Vienna meeting of the Astronomische Gesellschaft it was voted to solicit funds for the erection of a Gauss tower on Hohenhagen, the highest mountain near Göttingen, at the vertex of the Brocken-Göttingen-Hohenhagen triangle with which Gauss experimented concerning non-euclidean geometry. Ten thousand Marks have already been received. It is desired that this sum may be sufficiently increased to warrant the laying the corner stone April 30, 1909, the birthday of Gauss. Contributions may be sent to Professor F. Klein, Göttingen, Wilhelm Weberstrasse 3.

THE reproduction of an unpublished photograph of DIRICHLET, superior to any heretofore obtainable, will be issued, provided enough subscriptions are received to defray the expense. The size will be 14 × 18 cm. and the price is two Marks. Orders may be sent to Frl. Lotte Nelson, Darmstadt, Moosbergstrasse 43.

DR. F. RUSL, docent in mathematics at the Bohemian University of Prague, has been appointed associate professor of mathematics at the Bohemian technical school of Prague.

PROFESSOR R. RAU, of the University of Jena, will retire from his university position at the close of the present semester.

Professor M. Delassus, of the University of Besançon, has accepted the professorship of mechanics at the University of Bordeaux.

Dr. G. D. Birkhoff, of the University of Wisconsin, has accepted an assistant professorship of mathematics at Princeton University.

Professor Otto Rupp, of the technical school at Brünn, died in January, at the age of 54 years.

Mr. Martin Schilling, manufacturer of mathematical models and apparatus at Halle, died December 24, 1908, at the age of 42 years.

Professor E. A. Legoux, of the University of Toulouse, died January 6, at the age of 67 years.

Professor Hermann Minkowski, of the University of Göttingen, died January 12. He was born at Alexoten, Russia, June 22, 1864, studied at Königsberg and Berlin, held associate professorships of mathematics at Bonn and Königsberg, and was promoted to a full professorship at Königsberg in 1895. The next year he was called to the technical school at Zürich; he accepted the newly established professorship of mathematics at Göttingen in 1903.

Recent catalogues of second-hand mathematical books:— Bowes and Bowes, 1 Trinity Street, Cambridge, England, catalogue No. 326, later period, about 2600 titles.—A. Hermann, 6 rue de la Sorbonne, Paris, catalogue No. 95, 1900 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Becker (J.). Die Riccatische Differentialgleichung. (Progr.) Karlsbad, 1908. 8vo. 25 pp.

Börner (E.). Beweis eines Satzes aus der Gruppentheorie. (Progr.) Wien, 1908. 8vo. 13 pp.

Bücher, neue, über Naturwissenschaften und Mathematik. Mitgeteilt Herbst 1908. Leipzig, Hinrich, 8vo. 19 pp. M. 0.30

Czuber (W.). Einführung in die höhere Mathematik. Leipzig, Teubner, 1909. 8vo. 10 + 382 pp. Cloth. M. 12.00.

Dieck (W.). Zur Klassifikation der Punktepaar- und Kegelschnitt-Büschel. (Progr.) Sterkrade, 1908. 8vo. 40 pp.

Dintzl (E.). Einführung in die Funktionenlehre für Schüler der oberen Klassen an Mittelschulen. (Progr.) Wien, 1908. 8vo. 49 pp.

García Antúnez (C.). Complemento del cálculo infinitesimal. Procedimientos geométricos y aparatos de integración. Guadalajara, Ramírez, 1908. 120 pp. P. 5.00

Geiger (K.). Dreiteilung eines Winkels a) mit Hilfe einer Kurve dritter Ordnung, b) mittels einer Hyperbel, deren Asymptotenwinkel 120° beträgt, und c) der geometrische Beweis der Unmöglichkeit einer elementaren Lösung. (Progr.) Landshut, 1908. 8vo. 25 pp.

Grohmann (A.). Die verschiedenen Zahlensysteme und Umwandlung von Zahlen in verschiedene Systeme. (Progr.) Wien, 1907. 8vo. 14 pp.

Haller (S.). Ueber die Brennpunktskurven von Kegelschnittsystemen. (Progr.) München, 1908. 8vo. 35 pp.

Harnack (A.). See Serret (J. A.).

Hess (W.). Beweis des grossen Fermatschen Satzes für Ungerades $n > 1$. Dresden, Köhler, 1908. 8vo. 4 pp. M. 0.50

——. Weiteres über den grossen Fermatschen Satz. Dresden, Kohler, 1908. 8vo. 4 pp. M. 0.50

Hinrichs (J.). Aufgaben aus der analytischen Geometrie für die Prima. (Progr.) Neustrelitz, 1908. 8vo. 55 pp.

Klobasa (C.). Ueber Pythagoräische und Heronische Zahlen. (Progr.) Troppau, 1908. 8vo. 20 pp.

Klug (R.). Des kaiserlichen Mathematikers Johannes Kepler Neujahrsgeschenk oder über die Sechseckform des Schnees. 1611. Aus dem lateinischen übersetzt und mit Anmerkungen versehen. (Progr.) Linz, 1907. 8vo. 30 pp.

Kowalewski (G.). Grundzüge der Differential- und Integralrechnung. Leipzig, Teubner, 1909. 8vo. 6 + 452 pp. Cloth. M. 12.00

Morgenstern (A.). Beiträge zur numerischen Lösung der Gleichungen fünften Grades. (Progr.) Berlin, 1908. 8vo. 47 pp.

Pasch (M.). Grundlagen der Analysis. Ausgearbeitet unter Mitwirkung von C. Thaer. Leipzig, Teubner, 1909. 8vo. 140 pp. M. 4.00

Renner (J.). Die Theorie der Abbildung. (Progr.) Graz, 1908. 8vo. 50 pp.

Rollner (F.). Ueber vereinfachte Grundlegung und Systematik der Geometrie. (Progr.) Iglau, 1908. 8vo. 11 pp.

Saxelby (F. M.). An introduction to practical mathematics. New York, Longmans, 1908. 12mo. 6 + 220 pp. Cloth. $0.80

Scheffers (G.) See Serret (J. A.).

Serret (J. A.). Lehrbuch der Differential- und Integralrechnung. Nach Axel Harnack's Uebersetzung, 4te und 5te Auflage, bearbeitet von G. Scheffers. Vol. I: Differentialrechnung. Leipzig, Teubner, 190 . 8vo. 16 + 626 pp. Cloth. M. 13.00

Thaer (C.). See Pasch (M.).

Weigelin (G.). Der Fermat-Satz und sein Beweis. 2ter Teil. Stuttgart, Enderlen, 1908. 8vo. 16 pp. M. 1.50

Weinfeld (K.). Zur Konstruktion des oskulierenden Hyperboloides an windschiefe Regelflächen. (Progr.) Jägerndorf, 1908. 8vo. 16 pp.

Wieleitner (H.). Spezielle ebene Kurven. (Sammlung Schubert, LVI.) Leipzig, Göschen, 1908. 8vo. 16 + 409 pp. Cloth. M. 12.00

Winkler (J.). Die Schwingungsfiguren in analytischer Behandlung. (Progr.) Cilli, 1908. 8vo. 32 pp.

II. ELEMENTARY MATHEMATICS.

Baudoin (P.). See Bourlet (C.).

Bertrand (J.) et Garcet (H.). Traité d'algèbre. 1re partie à l'usage des classes de mathématiques élémentaires; 2e partie à l'usage des classes de mathématiques spéciales. Nouvelle édition. Paris, Hachette, 1908. 8vo. 2 vols. 1re partie, 4 + 326 pp.; 2e partie, 392 pp. Fr. 10.00

Borel (E.) et Royer (M.). Algèbre à l'usage des écoles normales, primaires (programmes de 1905) des écoles primaires supérieures, des aspirants et aspirantes aux brevets de l'enseignement primaire et des lycées et collèges de jeunes filles. Paris, Colin, 1908. 16mo. 8 + 469 pp. Fr. 3.50

Bourlet (C.). Cours abrégé de géométrie dans l'espace, rédigé conformément aux programmes de 1905. Corrigés des 344 exercices et problèmes; avec la collaboration de P. Baudoin. Paris, Hachette, 1908. 16mo. 176 pp. Fr. 2.00

Bouvart (C.) et Ratinet (A.). Nouvelles tables de logarithmes à cinq décimales. Table numérique. Table trigonométrique. Division centésimale établie conformément à l'arrêté ministériel du 3 août 1901, à l'usage des candidats au baccalauréat et aux écoles polytechnique et de Saint-Cyr. 5e édition. Paris, Hachette, 1908. 8vo. 128 pp. Fr. 2.00

Coquelet (G.). Mathématiques spéciales. Concours d'admission à l'école polytechnique, à l'école centrale des arts et manufactures, à l'école des ponts et chaussées, etc. Préface de J. J. Pillet. Edition complète. Paris, Delagrave, 1909. 8vo. 215 pp. Fr. 5.00

Cracknell (A. G.). See Workman (W. P.).

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Frontera (D. J.). See Sonnet (H.).

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Gómez Pallete (J.). Trigonometría, rectilinea y esférica, considerada como introducción al estudio de la geometría analítica. 11a edición, aumentada con una escogida collección de ejercicios mineros. Madrid, Perlado, 1908. 144 pp. P. 5.00

Heis (E.). Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra zum Gebrauche an höheren Schulen. 112te Auflage. Nach den zeitgemässen Anforderungen des mathematischen Unterrichts neubearbeitet und erweitert von J. Druxes. 2ter Teil. Pensum der Ober-Sekunda, Unter- und Ober-Prima. Köln, Du Mont-Schauberg, 1908. 8vo. 7 + 273 pp. M. 2.80

Lieber (H.). See Müsebeck (C.).

Müsebeck (C.). Aufgaben für den Unterricht in der Stereometrie und sphärischen Trigonometrie, enthaltend Aufgaben zu Teil III des Leitfadens der Elementar-Mathematik von H. Lieber und F. von Lühmann. Berlin, Simion, 1908. 8vo. 6 + 106 pp. M. 1.80

Pillet (J. J.). See Coquelet (G.).

BATINET (A.). See BOUVART (C.).

ROYER (M.). See BOREL (E.).

SCHMEHL (C.). Sammlung von Aufgaben aus der Algebra und algebraischen Analysis. Fur die Prima der realistischen Anstalten. Auflösungen. Giessen, Roth, 1909. 8vo. 69 pp. M. 1.80

SCHUBERT (H.). Beispiel-Sammlung zur Arithmetik und Algebra. 8te, durchgesehene Auflage. Neudruck. (Sammlung Göschen, 48.) Leipzig, Göschen, 1908. 16mo. 147 pp. Cloth. M. 0.80

SMITH (D. E.). Rara arithmetica. Boston, Ginn, 1908. 8vo. xvi + 506 pp. Edition de luxe. 2 vols. Vellum. $20.00

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THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and forty-second regular meeting of the Society was held in New York City on Saturday, February 27, 1909. A single session sufficed for the brief programme. The attendance included the following twenty-four members:

Professor Maxime Bôcher, Professor E. W. Brown, Professor F. N. Cole, Miss E. B. Cowley, Miss L. E. Cummings, Professor H. B. Fine, Dr. Frank Irwin, Professor Edward Kasner, Mr. W. C. Krathwohl, Mr. H. F. MacNeish, Professor W. F. Osgood, Professor R. W. Prentiss, Mr. H. W. Reddick, Professor L. W. Reid, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Dr. Elijah Swift, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. C. B. Walsh, Professor H. S. White, Miss E. C. Williams.

The President of the Society, Professor Maxime Bôcher, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. W. T. Campbell, Boston Latin School; Professor W. A. Garrison, Union College; Mr. D. D. Leib, Johns Hopkins University; Professor William Marshall, Purdue University; Mr. J. B. Smith, Richmond, Va., High School; Mr. C. M. Sparrow, Johns Hopkins University. Ten applications for membership in the Society were received.

Professor Bôcher presented his resignation as member of the Editorial Committee of the *Transactions*, to take effect August 15, it being his intention to spend the coming year abroad. Professor Osgood was appointed to fill Professor Bôcher's unexpired term.

The following papers were read at this meeting:

(1) Professor EDWARD KASNER: "Brachistochrones and tautochrones."

(2) Dr. D. C. GILLESPIE: "On extremal curves which are invariant under a continuous point transformation group."

(3) Professor VIRGIL SNYDER: "Infinite discontinuous groups of birational transformations which leave certain surfaces invariant."

Professor Bôcher presented certain results in extension of his paper "On systems of linear differential equations of the first order," read February 22, 1902, and to be published in an early number of the *Transactions*.

In the absence of the authors the papers of Dr. Gillespie and Professor Snyder were read by title. Professor Kasner's paper will appear in the BULLETIN. Abstracts of the other papers follow below, being numbered to correspond to the titles in the list above.

2. Dr. Gillespie studies the connection between an integral, its extremal curves, and a continuous group of point transformations under which they may be invariant.

The condition that a two-parameter family of curves in the plane be the extremal curves of an integral is expressed by a partial differential equation of the first order (a), which the integrand must satisfy.* This system of extremals is the solution of an ordinary differential equation of the second order (b). If the extremals are invariant under a transformation group, there exists at least one first integral of the equation (b) which is invariant under the transformation; *i. e.*, every one parameter family of curves obtained by assigning a value to the arbitrary constant in this first integral is invariant under the transformation.† This first integral together with the transformation defines a solution of the equation (a) which contains an arbitrary function, but which is not necessarily the complete solution of the equation. The relation between this first integral, the transformation, and the system of solutions of (a) can be expressed by an equation. When the first integral is one of the knowns, this is an algebraic equation; when the first integral is the unknown, it is a differential equation, which requires nevertheless only a quadrature for its solution.

3. Only a few isolated examples of surfaces have been found which are invariant under a discontinuous group of birational transformations of infinite order. Professor Snyder shows that these examples all depend upon a certain (2, 2) correspondence. Three interpretations are given. The first applies to quartic surfaces having more than one conical point. In any plane

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* Bolza, Lectures on the calculus of variations, p. 31.

† Lie-Scheffers, Vorlesungen über Differentialgleichungen, p. 375. Page, Ordinary differential equations, p. 149.

through two conical points the (2, 2) correspondence is defined by the pencils through the nodes. The second also refers to quartic surfaces, those having two nets of hyperelliptic curves. The (2, 2) correspondence is defined by the lines joining the points of the canonical g_2^1. The third concerns the systems of bitangents on any surface which is complete focal surface of two or more congruences.

F. N. COLE,
Secretary.

BÉZOUT'S THEORY OF RESULTANTS AND ITS INFLUENCE ON GEOMETRY.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY DECEMBER 30, 1908.

BY PRESIDENT HENRY S. WHITE.

THE accepted truths of today, even the commonplace truths of any science, were the doubtful or the novel theories of yesterday. Some indeed of prime importance were long esteemed of slight importance and almost forgotten. The first effect of reading in the history of science is a naive astonishment at the darkness of past centuries, but the ultimate effect is a fervent admiration for the progress achieved by former generations, for the triumphs of persistence and of genius. The easy credulity with which a young student supposes that of course every algebraic equation must have a root gives place finally to a delight in the slow conquest of the realm of imaginary numbers, and in the youthful genius of a Gauss who could demonstrate this once obscure fundamental proposition.

The first complete proof, by Gauss, that rational algebraic equations have roots either real or imaginary dates back only to 1799. That part of algebra that is concerned with equations is accordingly for the most part modern, recent indeed, as compared for instance with the plane geometry of lines and circles. Before Gauss, it is true, much had been done in actually solving equations of the lower orders and in the theory of symmetric functions of the roots. After him also the question of the arithmetical character of the roots required not only a Galois to penetrate its mystery, but also a Liouville and a Jordan to expound the marvellous theory that Galois had created.

In the general mathematical consciousness of the present day is dominant, even when not active, the knowledge revealed by Gauss and Galois; and this widespread understanding of the nature of the solutions of a single equation constitutes the goal toward which tended for three centuries the labors of Tartaglia, Cardan, Ferrari, and the long train of emulous scholars culminating in Lagrange and Abel. The theory of the single equation in one unknown is now, if not completed in all its details, yet at least finished in foundation and in the framework of its superstructure.

Equations in two variables have received no less attention, but from the nature of the case it must be long before their theory could reach any corresponding stage of completeness. Geometrically, a set of discrete points on a line has obviously fewer properties to investigate than a curve in a plane. The projective covariants of the one are all, like itself, sets of discrete points; while those of the other are of four or more different kinds: points, lines, curves, and line-loci or envelopes. Of a binary form, or an ordinary equation in one unknown, one seems to have a satisfactory picture in a set of a definite number of points on a line; but in endeavoring to apprehend even the picture of a plane curve, one begins instinctively to look for inflexional tangents, bitangents, and soon for polars, Hessian, Cayleyan, and other auxiliary loci; so that an algebraic curve becomes to our understanding an associated host of curves, even before we rise to the concept of a class of Riemann surfaces. And in fact, by reflex influence, since the developments of the past forty or fifty years, the geometry of sets of points on a line is now concerned less with the individual set, but much more with the related infinite linear systems or involutions.

Hence simultaneous equations were necessarily to be discussed, if geometry was to develop in the algebraic direction foreshadowed by Descartes and Newton. Descartes it was who proposed to classify plane curves as algebraic and not algebraic, and to divide the former according to the degree of their equations. Analysis quickly overtook pure geometry in developing the properties of loci of the second order, and it was easy for Euler to prove that the equation for the intersection of two second order loci is of degree 4. The next step in generalization was to determine the degree of the eliminant, or equation for the common points satisfying two simultaneous equations of any orders, m and n. This question was solved independently in 1764 by Euler, whose interest in mathematics was universal, and by a young French student who seems to have confined himself exclusively to this narrow field of research, Etienne Bézout. Both gave the degree as $m \cdot n$, the product of the orders of the intersecting loci, and both proved the theorem by reducing the problem to one of elimination from an auxiliary set of linear equations. Both, that is, depended upon the formal structure of what were later named *determinants*. The result of this initial publication, restricted to two equations, is what has kept current the fame of Bézout, and the determinant resulting from his method is what Sylvester and later writers call the Bézoutiant. But this was to Bézout only the beginning of his lifelong study in the formation of eliminants.

The mode of formation of this resultant of two equations in one unknown, or eliminant of two in two variables, is familiar to most students; Brill and Noether show that an equivalent process was employed by Newton for forming resultants of equations of low degrees, though without the almost obvious extension to equations in two unknowns. Two things about it were patent as clues to generalization. First, it gave a direct process for combining any two equations to eliminate any one unknown. Second, the resultant when found is a linear combination of the two original functions, with multipliers that are rational in the variables and in the coefficients of the given functions,

$$R \equiv F_1 f_1 + F_2 f_2.$$

With more equations and more variables, which of these two features would be more useful? We must remember that in 1765 not even the degree of the eliminant was known, and that the chief effect desired in a scheme for removing two or more unknowns from a set of equations was that it should indicate the degree of the result (e. g., the exact number of intersections of three surfaces of given orders).

The use of a direct process, to be applied step by step in some systematic sequence, is certainly the more tempting; and this mode has been employed very extensively of late — the highest common factor process — as affording more cogent deductive arguments at every stage. But it has its difficulties, as Bézout indeed found out; for he experimented many years before he found the more feasible plan. If one determines to use it, as does Kronecker for example, one must discriminate with care between an essential factor and an adventitious factor in every resultant, and learn to calculate in advance the degree of each. Bézout chose finally the other horn of the dilemma, and availed himself of the right to use hypothesis; he ventured to guess that the other style of attack would be successful.

It was his hypothesis that from $k+1$ equations k variables could be eliminated at one stroke; and that the result would be a linear combination of the given functions. This latter is equivalent to the famous Fundamental-Satz of Noether, and its proof is usually based upon the existence and the particular degree of the resultant. So we recognize the venturesome character of this attack. Supposing, then, that by using multipliers F_i of degree sufficiently high he could produce a combination

$$R_n \equiv F_{m_1} f_{n_1} + F_{m_2} f_{n_2} + \cdots + F_{m_k} f_{n_k} + F_{m_{k+1}} f_{n_{k+1}}$$

that should be free from k variables or unknowns, say

$$x_1,\ x_2,\ \cdots,\ x_k$$

and contain only x_{k+1}, he set out to determine the necessary value of n, the degree of R in the sole remaining unknown x_{k+1}. That is to say, he sought for a minimum value n which should be sufficiently great for the purpose.

Naturally his work has received many improvements from later workers; and we must recognize the large credit due to Professor Eugen Netto for having worked out so many supplementary lemmas in his admirable Vorlesungen über Algebra. To rescue a theory from its own weaknesses is no less deserving of honor than the invention of new theories. Only by a vast amount of such patient self-forgetful devotion to the interests of science can the body of demonstrated truth grow as a solid and unshakable structure. It will not injuriously misrepresent Bézout therefore if we say that he considered all the coefficients in the multipliers $F_{m_1}, F_{m_2}, \cdots, F_{m_{k+1}}$ as undetermined arbitrary parameters, and inquired first *how many of them must remain arbitrary after the resultant R_n has been completely determined.* Obviously any two terms in the combination could be modified together, since identically

$$F_a f_a + F_\beta f_b \equiv (F_a + \phi_{a,b} \cdot f_b) f_a + (F_\beta - \phi_{a,b}\, f_a) f_b,$$

so that the coefficients in all such functions $\phi_{a,b}$ will remain *essentially indeterminate.* But again, in this way certain modify-ing terms are enumerated more than once (as those containing the product $f_a \cdot f_b \cdot f_c$, etc.). Hence was necessary a considerable invention and application of a calculus of finite differences to solve the enumerative problem: this forms the first part of Bézout's great work; and this is reproduced well by Serret (Algèbre supérieure) and Netto.

Excluding essential indeterminates, call the number of *available* undetermined parameters P, or say $P(n, k+1)$; and let $N(n, k+1)$ denote the number of terms that can occur in a polynomial of order n in $k+1$ variables. Since $n+1$ terms in x_{k+1} are expected in R, the number to be removed is $N(n, k+1) - n$. Hence the inequality to be satisfied, stating that there are no more terms to be excluded than there are available arbitrary multipliers, is

$$P(n, n_1, n_2, \cdots, k+1) - 1 \geqq N(n, k+1) - n - 1$$

or

$$N - P \leqq n.$$

If R is to be a determinate function, this shows that its degree must be exactly $N - P$. But as n increases from the largest of the numbers $n_1, n_2, \cdots, n_{k+1}$, $N - P$ reaches soon its maximum value and then, as a constant, remains superior in magnitude until n becomes equal to the product $n_1 \cdot n_2 \cdot n_3 \cdots n_{k+1}$. From that point on of course the inequality is reversed; that is, the parameters are more numerous than the conditions to be satisfied. Hence *the degree of any completely determinate resultant must be exactly the product of the degrees of the original equations.*

This leaves unsettled of course first the question of the *existence* of such a resultant, namely, a determinate linear combination of the given functions with rational multipliers; in other words the question whether for $n = N - P$ the linear equations to be satisfied are all independent and consistent. Also, secondly, it leaves to be fixed by further consideration the choice of terms whose coefficients shall be equated to zero, in order from the linear equations thus formed to eliminate the auxiliary coefficients by means already familiar and so to obtain the eliminant expressed as a determinant. On this second point no one has yet succeeded in elaborating such a scheme for substitution as evidently Bézout supposed was practicable. We shall find his own words on this point not critically precise, but interesting as showing how loosely constructed arguments once passed current. "But if we imagine that the first equation is multiplied by a complete polynomial of order m in the same number of unknowns, and that in the resulting equation of degree $m_1 + n_1$ (the 'product equation') there are substituted in all the terms where it is possible to do this the value of $x_1^{n_2}$, that of $x_2^{n_3}$, that of $x_3^{n_4}$, etc., then as the multiplier polynomials will have brought into the product equation as many different coefficients as there are terms, we see that after those substitutions there can remain of terms in x_1, x_2, x_3, etc., only so many as it will be possible to make disappear by the help of the coefficients in the polynomial multipliers." Further on, more to the same effect about these substitutions, but nowhere an examination into the difficulties of such substitutions when more than one quantity is to have its exponent reduced.

No such difficulty arises if only a very special set of equations is considered, namely, that in which each equation in its turn contains only the first power of its correspondingly numbered unknown quantity together with other terms involving later unknowns but none earlier,

$$\begin{aligned} f_{n_1} &= x_1 - g_{n_1}(x_2, x_3, x_4, \cdots, x_{k+1}) = 0, \\ f_{n_2} &= x_2 - g_{n_2}(x_3, x_4, \cdots, x_{k+1}) = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_{n_k} &= x_k - g_k(x_{k+1}) = 0, \end{aligned}$$

while the last may be of any form $f_{n_{k+1}}(x_1, x_2, \cdots, x_k, x_{k+1}) = 0$. What Bézout very properly terms substitution is described more fully by precisians of the present century as the addition of the function f_1 multiplied by a suitable polynomial; since a term like ax_1^r for instance is replaced by adding to it the product

$$a(x_1 - g_{n_1}) \cdot \left\{ \frac{x_1^r - g_{n_1}^r}{x_1 - g_{n_1}} \right\} \equiv a(x_1^r - g_{n_1}^r).$$

In this example, an unknown once removed from $f_{n_{k+1}}(x_1, x_2, \cdots, x_k, x_{k+1})$ remains absent, and the first k of them being removed thus from the $(k + 1)$th, there remains the eliminant in the single unknown x_{k+1} to the proper degree.

But for the general "complete" equations, the attempt to construct the elimination process *progressively* after this model has not met with success hitherto, since an unknown after having its degree lowered by an early substitution may have it raised by a later substitution. Even Netto here attempts nothing further than a demonstration by the aid of linear equations. The general case of course would be not the very simple example cited above; each variable would appear in each to several different powers, and direct substitution would lower the degree of one variable only; the aim being to depress the degree to a maximum $n_r - 1$ for x_r $(r < k + 1)$. If this is possible, it gives what may be called for certain purposes a normal form.

Now this is a topic of scarcely less interest and importance than the matter of elimination itself — the reduction of a polynomial, by the aid of a set of equations, to another form in which the variables shall appear with exponents not higher than prescribed limiting values. This is for Bézout a transition form, from which finally the coefficients of all terms containing the variables to be removed are set equal to zero, giving the linear system needed for the closing step of his process of elimination. But as the form is unique, it should prove and has proved equally interesting, as a reduced normal form in a modular system, with the other reduced normal form that we call *the* eliminant.

There were still questions to be investigated when Bézout laid down his pen and published his *magnum opus* in 1779; but certainly he did not leave the subject where he found it. Where he left it, Cayley was to resume it later and at one stroke express a resultant not indeed by a single determinant, but by a fraction whose numerator and denominator contain only determinant factors, found rationally from the coefficients of the given equations. Bézout's system of equations is made to yield a definitive result when once the distinction of available and non-available parameters is abrogated, and all are treated alike. And as for the *degree* of the eliminant, the main problem in his time, Bézout had fixed that both for the so-called general case, and also for a large number of particular cases. One may say that he determined the number of finite intersections of algebraic loci, not only when all the intersections are finite, but also when singular points, or singular lines, planes, etc., at infinity occasion the withdrawal to infinity of certain of the intersection points; and this at a time when the nature of such singularities had not been developed.

All algebraic geometry is a reduction, actual or possible, of systems of equations to a different form, plus the specification of what the symbols shall denote. The resultant being once known, of course its interpretation will thereafter be found often in geometric discussions. To survey the applications of the eliminant therefore is practically to enumerate the different chapters that have been created in algebraic geometry.

The first would certainly be what is called Cramer's paradox. This antedates Bézout, for Cramer's book appeared in 1750, and may probably have helped to form Bézout's opinion on the urgent need for a solution of his problem. The paradox is, briefly, this. The equation of a plane curve of order n contains $\frac{1}{2}(n+1)(n+2)$ terms, depends therefore on the same number of coefficients, less one. To require the curve to pass through that number of given points is then, algebraically speaking, to determine the coefficients by means of linear equations, hence *uniquely*. But two curves of that order have in common more than that number of points, namely, n^2 (according to the Bézout theorem). But $n^2 > \frac{1}{2}(n^2+3n)$ if $n > 3$, so that two curves, indeed an infinite pencil of curves of order n can be passed through even more points than would commonly suffice to determine a single curve. This not intricate puzzle was resolved by Euler, leading to the interesting and important notion of interdependent sets of points in a plane. From this concept was to arise later that of involutions of points in a plane (or in space of more than two dimensions), and thus was opened a field which has only been entered, but not yet explored. An involution is generated when in a set of points of intersection of two (or three) curves enough points are fixed to leave only one still arbitrarily variable, while $r-1$ others vary with it. This one then being supposed to describe the whole plane, the totality of positions of the whole set of r varying points, a doubly infinite system of point sets, constitutes an *involution* in the plane.

Next in order of development is the notion of multiple points of curves. Bézout treated them by implication only, locating them at infinity, and showed for many cases how they affect the number of finite intersections. These questions were taken up, five decades later, by Plücker, and led to the discovery of the universally known Plücker relations that connect the numbers of double points, inflexional points, double tangents, and cusps, of a plane curve. These latter gave to geometry the notion of the *deficiency* of a curve, that notion which was to be analyzed to its depths by Riemann and by Clebsch, and to form the chief organic bond between algebraic curves and the theory of multiply periodic functions of a complex variable.

But Cramer's paradox led naturally also to the examination of the sets of variable intersections cut out upon a curve of one order by a varying curve of different order; that is, to the questions of curves having contacts one or many on the fixed curve; of the degree of freedom in a variable point set on the curve; in short to all those questions which lead through Noether's Fundamental-theorem. Before Noether there were theorems in plenty upon the partition of the intersections of two curves into two sets, with the hypothesis that one partial set lie all on one curve of lower order, and the conclusion that the remaining partial set must lie on another curve of complementary order. The best known of these are ascribed to Plücker, Jacobi, and Cayley. But Noether was the first to formulate precisely and prove what was tacitly assumed in them all, that a curve containing all the intersections of two others, $f_1 = 0$ and $f_2 = 0$, can always be represented in the form $F_1 f_1 + F_2 f_2 = 0$, where F_1 and F_2 are rational in the coordinates. One sees that this form is that in which Bézout assumed the resultant, and which he in effect established. Very naturally therefore Noether's proof begins upon that basis, and lays down conditions which while not demonstrably *necessary*, are certainly *sufficient* for the conclusion. Upon this basis then rises the extensive body of theorems relating to adjoint and non-adjoint curves, and in particular the elegant theorem on residual and corresidual point sets on a curve: *If a point set A is corresidual with another B in respect to any third set C, then any fourth set residual to A is residual also to B;* two residual sets constituting, together with the singular points of the basal curve, its complete intersection with an adjoint curve.

Bézout's discussions had regard altogether to the elimination from numerical equations; for in his day the theory of forms, of invariants under the projective group, was not yet known. When that theory arose, the interest of algebra in the resultant was no less keen than that of geometry in the eliminant. Hence in the plane the result of eliminating both coordinates from three equations was examined, an invariant whose vanishing gives the condition for three curves to have a point in common. But three curves might have two, or three, or more, points in common. Cayley therefore proposed the question: If all the first polars of a curve have d points in common, what will be the degree, in the coefficients, of the condition for the occurrence of one further intersection? His answer was $3(n-1)^2 - 7d$, n

being the order of the given curve. More general was Brill's inquiry: When three curves of orders n_1, n_2, n_3 have d points of intersection, what is the condition for the occurrence of one further common point? This question he answers; and so was instituted the theory, yet in its infancy, of *reduced discriminants* and *reduced resultants*.

Pausing briefly in this review of the potentialities of the theorem on the degree of the resultant, though we have spoken so far only of resultants of two forms, let us observe Bézout's own view of the importance of his investigations. In the dedication he says of the work: "It has as its aim the completion of one part of the science of mathematics, on which all the other parts are waiting for that which can at this time secure their own advancement." Then in his preface, after deriding those who had been discouraged from researches barely begun by the complexity of the algebraic relations which they encountered, he continues as follows. "The analysis of infinitesimals, equally attractive and important, . . . has drawn away all the interest and all the toil, and the algebraic analysis of finite quantities, to start from that epoch, seems to have been looked at only as a field in which either there remained nothing further to be done, or else whatever was left to do would have proved fruitless speculation. . . . If we note carefully the fact that in reference to the countless number of equations and of unknowns, upon which the solution of any problem may depend, we know as yet how to treat only the case of two equations and two unknowns; that we understand, I repeat, how to treat only that single case with the certainty of introducing nothing extraneous to the question, then we shall doubtless agree that in this matter everything is yet to be done."

After explaining his methods in 463 pages, Bézout sums up the outcome thus (page xxi). "We think it possible to state that there is no kind of algebraic equations for which we have not given the means of determining the lowest possible degree of the final equation; either when there are or when there are not relations between the coefficients which could occasion a particular lowering of that degree." This belief may be a trifle optimistic, at least it is to be hoped that there will be found means of reaching *enumerative* results with less labor than Bézout's methods would impose. But surely the aim was high, and worthy the devotion of one life time; and the conclusion of his preface must certainly excite our admiration and respect.

"We hope that this work may prove the occasion of great progress in analysis, by turning toward that important field the talent and the cleverness of analysts of our time. We shall regard ourselves as fortunate if on considering the point where we take up these questions and the point where we leave them, it shall be found that we have discharged some part of that tribute which every man owes to society in that state in which he happens to be placed."

A life of unremitting labor is not ill spent if it leaves a work so easily intelligible, so full of interesting problems, and in proportion to contemporary science so complete as this Théorie générale des équations algébriques of Bézout. Yet what a commentary on the futility of the best efforts is found in the fact that both Jacobi and Minding, only 60 years later, published investigations as new whose methods and results were in effect identical with Bézout's! At least this showed not that his work was unnecessary, but only that he was in advance of his time. Was it perhaps that geometry was waiting for Gauss to prove that the degree of an equation indicates actual and not illusory roots? Or was it waiting for Monge and Poncelet and Plücker to set it free from the restraints imposed by a particular system of coordinates? Or perhaps for Liouville or Poisson to adjoin the linear function of all the coordinates with as many indeterminates: $z = k_1x_1 + k_2x_2 + \cdots + k_{k+1}x_{k+1}$, and to discuss not the resultant or eliminant in a single x_v, but the eliminant in z containing all the indeterminates $k_1, k_2, \cdots, k_{k+1}$? The algebraic projective geometry of the present has grown and flourished upon all these preparations; and it appears that not the single isolated worker but the cooperation of a large number, with the utmost facility of communication of results, is needful for the rapid advancement of any branch of mathematics.

To return to the geometrical problems that grow out of the theory of eliminants. In three dimensions, three surfaces intersect in points, real or imaginary, whose number is the product of their three orders. Three quadric surfaces meet in 8 points. Here, as in the plane, arises the question of the interdependence of these points: Cramer's paradox is applicable to space of any number of dimensions, and indeed to intersection systems which are not all points. Of these 8 intersections of three quadrics, if seven are given the eighth can be constructed linearly; and it is surprising to see how many eminent geometers have found in this one problem points of difficulty worthy of their attention. But an eliminant in more than two dimensions, *i. e.*, an intersection system of three or more loci, has a further peculiarly interesting possibility.

Three surfaces may have not only a finite set of points in common, but equally well also a whole curve or system of curves. That is to say, their eliminant may vanish identically, having an infinite number of roots. Three quadric surfaces can have in common a line, two lines, a conic, three lines, or a twisted cubic curve. What would then be the number of additional discrete points of intersection? If there is a common conic *C*, for example, two of the surfaces meet in *C* and a second conic *K*, which have two points in common; and the third surface must cut *K* in 4 points, the two upon *C*, and therefore only *two* outside the common conic *C*. Or speaking briefly, a common conic lying on three quadric surfaces *absorbs* six of their eight intersections. Similarly, a common line absorbs 4, a twisted cubic curve all 8, of the points of intersection of three quadric surfaces. This line of problems is not without difficulty, and seems not to have come to the notice of Bézout except where the common curves were straight lines at infinity; and even here he does not put the description into geometric language.

Evidently there was need of a theory of twisted curves in three dimensions; of gauche surfaces and curves in four dimensions, etc. Cases were known in which not 3, but 4, surfaces were required to define an intersection curve. For this case however there was in Bézout's eliminant a suggestion of a useful mode of attack. By eliminating one variable from two equations one has as eliminant the equation of a cone with vertex at infinity. Treat this in homogeneous coordinates, and we have a cone with vertex at any point, and containing the curve in such a way that on each generator there lies a single point of the curve. A different way of conducting the elimination suggested to Cayley the surface called, since his invention, a monoid; and both Noether and Halphen found the cone-and-monoid a sufficient representative of any twisted curve in 3 dimensions. But of curves and surfaces in projective space of four or more dimensions, I believe that no one has yet worked out a systematic list.

It is to be expected that fundamental propositions will gradually become obscured by their derivatives. When once a theorem like Noether's has been discovered, which embodies the geometrically most available substance of the Bézout theorem, and when this has found expression in geometric propositions on point sets, residues, and groups of point sets on a curve, then it will be but infrequently that geometers will have recourse to the formation of resultants or eliminants. Yet it is not well that such a basal theorem together with the scheme of operation that it involves should again fall into oblivion. For even at this late date one may read in a quite recently issued American cyclopedia the statement that, in complicated equations, elimination becomes difficult and often impossible. And even in Cayley's article in the Encyclopædia Britannica one reads the more temperate verdict that there does not yet exist a distinct theory of systems of equations! Such statements in what purport to be standard popular works of reference are more often read and credited than the precise summaries given in technical works or even the new mathematical Encyklopädie.

Two relatively recent essays employing resultants and discriminants for the extension of the theory of curves are by Franz Meyer, in the *Mathematische Annalen*, volumes 38 and 43. They proceed from the postulate that truths purely numerical can be established for the kind of curves called rational, and will then hold true for non-rational curves of the same order. The first deals with the ordinary singularities of plane curves; the second, with those of twisted curves in space, determining what consequences ensue to other singularities when two of any one sort come to coincide: *e. g.*, how many inflexions come together as two contacts of a double tangent merge into a hyperosculation point. The immediate aim is to find relations among the real singularities as distinguished from the imaginary ones; but the postulate seems valid and susceptible of adaptation to other uses.

It is not proper to close this sketch without alluding to Kronecker's work in formulating a systematic theory of systems of equations. Under the head of *Modular Systems* he instituted inquiries far beyond the single question of eliminants which filled Bézout's horizon. Some time ago allusion was made to one reduced form of equal significance with the eliminant, where all the variables were retained, but with exponents so lowered that the reduced form was uniquely determined. The study of all such reduced forms under any given system of equations (or moduli), whether general or special, is part of Kronecker's programme, and in particular the grouping together of all forms, which, like the resultant, are reducible to zero by the aid of given equations, under the class name of an algebraic modulus. In his Festschrift and the later expository papers of his pupils are proposed methods for testing any given system for its character, whether general, or special of the first sort (loci with a curve in common), or of the second or higher sort (loci with a surface, etc., in common). The expansion of this body of doctrine or abstract theory into a concrete *geometry* with fulness of examples remains a task, not all deductive but largely creative, for coming decades or generations.

Not the possession of eliminants actually calculated by Bézout's deservedly famous scheme is needful for the geometer, but the knowledge of the conditions under which such an eliminant will exist, and what conditions will modify it. So with regard to the more far-reaching scheme of Kronecker; it is ultimately, perhaps, not the full elaboration of particular examples as such, that we wish to have, but a precise knowledge of *how* the relative operations could be executed in finite time, and a precise formulation of conditions that would modify or influence the result of those operations. Which is of greater value, the logic or the concrete object to which it is applied? Let everyone decide when both are in his possession!

ON THE REPRESENTATION OF NUMBERS BY MODULAR FORMS.

BY PROFESSOR L. E. DICKSON.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1909.)

1. FOR any field F in which there is an irreducible equation $f(\rho) = 0$ of degree m, the norm of

$$x_0 + x_1\rho + x_2\rho^2 + \cdots + x_{m-1}\rho^{m-1}$$

is a form of degree m in m variables which vanishes for no set of values x_i in the field F, other than the set in which every $x_i = 0$. For a finite field it seems to be true that every form of degree m in $m+1$ variables vanishes for values, not all zero, in the field. For $m = 2$ and $m = 3$ this theorem is established in §§ 2, 3. The corresponding theorem does not hold in general for infinite fields. But A. Meyer * has shown that any indefinite quadratic form in five variables vanishes for integral values, not all zero, of the variables.

Modular forms which represent only squares have been investigated † at length by the writer; those which represent cubes exclusively are considered in §§ 4–12.

2. Any ternary quadratic form in the $GF[p^n]$, $p > 2$, can be transformed into $q = ax^2 + by^2 + cz^2$. The number of sets of solutions of $q = 0$ in the field has been determined; ‡ in particular, this number exceeds unity. The latter fact may be proved very simply as follows. Since the proposition is evident when $a = 0$, we may take $a = -1$, $c \neq 0$. Let $z = ty$. Then the question is whether or not $(b + ct^2)y^2 - x^2$ vanishes in the field for x and y not both zero. If not, $b + ct^2$ would be a not-square for every t. But there are only $\frac{1}{2}(p^n - 1)$ not-squares and $\frac{1}{2}(p^n + 1)$ squares t^2, including zero. Hence not every one of the $\frac{1}{2}(p^n + 1)$ values of $b + ct^2$ is a not-square.

3. Let Q be a quaternary cubic form with coefficients in the $GF[p^n]$, $p > 3$, which vanishes for no set of values in the field for x, y, z, w, except the set $x = 0, \cdots, w = 0$. The coefficient of x^3 is not zero and may be taken to be unity. By adding to x a linear function of y, z, w, we may delete the terms x^2y, x^2z, x^2w. The coefficient of x may be transformed into $dy^2 + \lambda z^2 + \mu w^2$. Then Q becomes

$$Q_1 = x^3 + x(dy^2 + \lambda z^2 + \mu w^2) + gy^3 + Ay^2z + By^2w + Cyz^2 \\ + Dyw^2 + Eyzw + Fz^3 + Gz^2w + Hzw^2w + R^3.$$

Let $w = tz$. Then Q_1 becomes a ternary cubic form which may be identified with that on page 161 of volume 14 of the BULLETIN, by setting $a = 1$, $b = c = f = 0$,

$$e = \lambda + \mu t^2, \quad h = A + Bt, \quad k = C + Et + Dt^2, \\ l = F + Gt + Ht^2 + Rt^3.$$

Hence the relations at the top of page 165 must hold for every t. From $dh \neq 0$, we conclude that $d \neq 0$, $B = 0$. Then from the coefficient of t^3 in $eh + 3\,dl = 0$, we have $R = 0$. Hence Q_1 vanishes for $x = y = z = 0$, $w = 1$, contrary to hypothesis.

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* Bachmann, Zahlentheorie, IV, p. 266, p. 553.

† *Transactions Amer. Math. Society*, vol. 10 (1909), pp. 109–122.

‡ The writer's Linear Groups, pp. 47, 48.

4. In investigating forms which represent exclusively cubes* in the $GF[p^n]$, we shall assume that p^n is of the form $3k+1$. For if p^n-1 were prime to 3, every element $\neq 0$ of the field would equal a cube in the field, and the question would be without content.

Let $ax^d+\cdots$ represent cubes exclusively. In view of the values $x=1$, $y=0, \cdots$, a must be a cube. Hence we may assume that the coefficient of x^d is unity. Further d must be a multiple of 3. For, if d is prime to 3, the form represents the not-cube v^d when x equals a not-cube v, $y=0, \cdots$. *We may therefore restrict our attention to forms* $x^{3l}+\cdots$ *in the* $GF[p^n]$, $p^n=3k+1$.

5. Theorem. *No cubic form represents cubes exclusively.*

It suffices to prove the theorem for irreducible binary cubic forms C in the $GF[p^n]$, $p^n=3k+1$. Then $C(x, 1)=0$ has the roots ρ, ρ^{p^n}, $\rho^{p^{2n}}$ in the $GF[p^{3n}]$, so that

$$C=(x-\rho y)^t, \qquad t=1+p^n+p^{2n}.$$

If C represent only cubes, $C^{(p^n-1)/3}=1$. Hence the power $\frac{1}{3}(p^{3n}-1)$ of $x-\rho y$ equals unity, so that $x-\rho y$ is a cube in the $GF[p^{3n}]$ for every x and y, not both zero, in the $GF[p^n]$. Any non-vanishing element μ of the latter is a cube in the $GF[p^{3n}]$. Hence

$$\mu(a-\rho)/(b-\rho) \qquad (a \neq b)$$

furnishes $p^n(p^n-1)^2$ distinct cubes in the $GF[p^{3n}]$, a number exceeding the total number $\frac{1}{3}(p^{3n}-1)$ of the cubes.

6. In the $GF[p^n]$, $p^n=3k+1$, let $S(x, y)$ be a sextic representing only cubes. Since S is the product of $S(xy^{-1}, 1)$ by y^6, it suffices to require that $S(x, 1)$ shall represent only cubes.

First, let $p=2$, so that n is even, $n=2m$. Set

$$(1) \qquad S=\sum_{i=0}^{6} a_i x^i \qquad (a_6=1).$$

Now $S^{(4^m-1)/3}=1$ since S shall represent only cubes in the $GF[4^m]$. The exponent equals $\sum_{j=0}^{m-1} 4^j$. Hence, for every x in the field,

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* A non-vanishing element equal to the cube of an element of the field.

$$\prod_{j=0}^{m-1}\left(\sum_{i=0}^{6} a_i^{4^j} x^{i\cdot 4^j}\right)=1.$$

The powers of x may be reduced by $x^{4^m}=x$. Hence, for $1 \leqq e \leqq 4^m-1$,

$$(2) \qquad \Sigma a_{i_0} a_{i_1}^4 a_{i_2}^{4^2} \cdots a_{i_{m-1}}^{4^{m-1}} = 0,$$

the sum extending over all * integers $0 \leqq i \leqq 6$ for which

$$(3) \quad E = i_0 + 4i_1 + 4^2 i_2 + \cdots + 4^{m-1} i_{m-1} \equiv e \pmod{4^m-1}.$$

It is found to be advantageous to consider the higher values of e.

For $e = 4^m-1$, we have $E = 2e$ (whence each $i = 6$) or $E = e$. But

$$(4) \qquad 4^m - 1 = 3 + 3\cdot 4 + 3\cdot 4^2 + \cdots + 3\cdot 4^{m-1}.$$

Hence $E = e$ gives $i_0 \equiv 3 \pmod 4$, whence $i_0 = 3$, $i_1 = 3$; etc. Since $a_6 = 1$, (2) becomes

$$(5) \qquad 1 + a_3^{(4^m-1)/3} = 0.$$

For $e = 4^m - 2$, $E = 2(4^m-1)-1$ gives $i_0 = 5$, $i_j = 6 (j \geqq 1)$. For $E = e$, either $i_0 = 2$, $i_j = 3 (j \geqq 1)$, or $i_0 = 6$. In the latter case, $\frac{1}{4}(E-e)$ gives

$$1 + i_1 - 3 + 4(i_2-3) + 4^2(i_3-3) + \cdots + 4^{m-2}(i_{m-1}-3) = 0.$$

Thus $i_1 \equiv 2 \pmod 4$. For $i_1 = 2$, each $i_j = 3 (j \geqq 2)$. For $i_1 = 6$,

$$1 + i_2 - 3 + 4(i_3-3) + \cdots + 4^{m-3}(i_{m-1}-3) = 0.$$

Either $i_2 = 2$, $i_j = 3 (j \geqq 3)$, or $i_2 = 6$ and we proceed as before. Thus

$$(6) \qquad a_5 + a_2 a_3^{c_1} + a_2^4 a_3^{c_2} + a_2^{4^2} a_3^{c_3} + \cdots a^{4^{m-1}}_2 a_3^{c_m} = 0$$

$$[c_r = \tfrac{1}{3}(4^m - 4^r)].$$

Similarly, for † $e = 4^m - t$, $t = 3, 4, 6, 7, 8$, we get

$$(7) \qquad a_4 + a_1 a_3^{c_1} + a_5(a_2^4 a_3^{c_2} + a_2^{4^2} a_3^{c_3} + \cdots + a_2^{4^{m-1}} a_3^{c_m}) = 0,$$

$$(8) \qquad a_3 + a_0 a_3^{c_1} + a_4(a_2^4 a_3^{c_2} + a_2^{4^2} a_3^{c_3} + \cdots) = 0,$$

$$(9) \qquad \begin{aligned} a_1 + a_5^{21} + a_2(a_2^4 a_3^{c_2} + a_2^{4^2} a_3^{c_3} + \cdots) \\ + a_1^4 a_3^{c_2} + a_5^4(a_2^{4^2} a_3^{c_3} + a_2^{4^3} a_3^{c_4} + \cdots) = 0; \end{aligned}$$

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* Except the set in which each $i = 0$, when $e = 4^m - 1$.

† The result for $e = 4^m - 5$ is evidently the fourth power of (6).

$$(10)\quad a_0 + a_4 a_5^4 + a_1(a_2^4 a_3^{c_2} + \cdots) + a_5 a_1^4 a_3^{c_2} + a_5^5(a_2^{4^2} a_3^{c_3} + \cdots) = 0,$$

$$(11)\quad a_3 a_5^4 + a_0(a_2^4 a_3^{c_2} + \cdots) + a_4 a_1^4 a_3^{c_3} + a_4 a_5^4(a_2^{4^2} a_3^{c_3} + \cdots) = 0.$$

For $m = 1$ the last three conditions do not occur.

First, let $a_5 = 0$. Replacing x by $x + a_4^{1/4}y$, we have $a_5 = 0$, $a_4 = 0$. By (5), a_3 is a cube in the field. By multiplying y by $a_3^{-1/3}$, we have $a_3 = 1$. Then $a_1 = 0$ by (7). By (6),

$$a_2 + a_2^4 + a_2^{4^2} + \cdots + a_2^{4^{m-1}} = 0,$$

so that $a_2 = 0$ if $m = 1$. For $m > 1$, (9) and (10) give $a_2 = 0$, $a_0 = 0$, so that the sextic vanishes for $x = 0$. Hence $m = 1$, and

$$(12)\quad S = x^6 + x^3y^3 + y^6 = [x^3 + \sigma y^3]\,[x^3 + (\sigma + 1)y^3],$$

where $\sigma^2 \equiv \sigma + 1 \pmod 2$ defines the $GF[4]$.

Next, for $a_5 \neq 0$, we multiply y and make $a_5 = 1$. We multiply (6) by a_3 and add the fourth power of (6). We obtain

$$1 + a_2 + a_3 = a_2 a_3^{c_1+1} = a_2,$$

by (5). Hence $a_3 = 1$, so that (6) becomes

$$(6')\qquad 1 + a_2 + a_2^4 + a_2^{4^2} + \cdots + a_2^{4^{m-1}} = 0.$$

In view of the latter, (7) and (3) give

$$(7')\qquad a_1 = a_4 + a_2 + 1, \quad a_0 = 1 + a_4(a_2 + 1).$$

If $m = 1$, $a_2 = 1$ by (6′), so that *

$$S = x^6 + x^5y + a_4x^4y^2 + x^3y^3 + x^2y^4 + a_4xy^5 + y^6.$$

If $a_2^4 + a_4 = 1$, we have (13). If $a_4 = 0$, we interchange x with y and are led to the earlier case $a_5 = 0$. If $a_4 = 1$, we replace x by $x + \sigma y$, where $\sigma^2 + \sigma + 1 = 0$ and obtain a sextic S with $a_4 = 0$.

If $m > 1$, we apply (6′) to (9) and obtain

$$a_1 + 1 + a_2(a_2 + 1) + a_1^4 + a_2^4 + a_2 + 1 = 0.$$

Eliminating a_1 and a_0 by (7′), we get

$$(9')\qquad a_2^2 + a_2 = a_4 + a_4^4.$$

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* In the simple case $m = 1$, we may also proceed directly with (1), which must reduce to unity (the only cube) for every x.

Condition (10) likewise leads to (9′). But (11) gives

$$(11') \qquad a_2 + a_2^2 a_4 + a_4 + a_4^5 + a_2 a_4 = 0.$$

By (9′) and (11′), we get

$$a_2 = a_4 + a_4^2.$$

By (7′), $a_1 = a_4^2 + 1$, $a_0 = a_4^3 + a_4^2 + a_4 + 1$. But $a_3 = a_6 = 1$. Hence

$$(13) \qquad S = [x^2 + xy + (a_4 + 1)y^2]^3.$$

THEOREM. *In the* $GF\,[2^n]$, *n even, every binary sextic which represents only cubes is formally a perfect cube, except for the case* $n = 2$, *when there is an additional type* (12), *the product of two irreducible cubic forms.*

7. For $p^n = 3k + 1$, $p > 2$, we may remove the term x^5y from the binary sextic S. Hence it suffices to investigate

$$(14) \qquad S = z^6 + bz^4 + cz^3 + dz^2 + ez + f \qquad (f = \text{cube}).$$

For $p^n = 7$, the cubes are ± 1. We require that $S^2 \equiv 1$ for every z making $z^6 \equiv 1 \pmod 7$. First, let $f = -1$. Then

$$(bz^4 + cz^3 + dz^2 + ez)^2 \equiv 1$$

requires that

$$be + cd \equiv 0, \quad d^2 + 2ce \equiv 0, \quad de \equiv 0, \quad b^2 + e^2 \equiv 0,$$
$$bc \equiv 0, \quad c^2 + 2bd \equiv 1.$$

Hence $S = z^6 \pm z^3 - 1$. Multiplying z by ± 1, we get

$$(15) \quad z^6 + z^3 - 1 = (z^2 - z + 3)(z^2 - 2z - 2)(z^2 + 3z - 1).$$

Next, let $f = 1$. Then $(bz^4 + \cdots + 2)^2 = 1$ gives

$$be + cd \equiv d^2 + 2ce + 4b \equiv 2c + de \equiv b^2 + e^2 + 4d$$
$$\equiv bc + 2e \equiv c^2 + 2bd + 3 \equiv 0.$$

If $e \equiv 0$, then $c \equiv 0$, $d \equiv -2b^2$, $b^3 \equiv -1$. Multiplying z by $3b^{-1}$, we have $b = 3$. Hence

$$(16) \qquad S = (z^2 + 1)^3.$$

If $e \neq 0$ we multiply y by e and have $e = 1$. Then the first three conditions give

$$b \equiv 2c^2, \quad c^2 \equiv c, \quad d \equiv 5c.$$

The fifth gives $c^3 \equiv -1$, in contradiction with $c^2 \equiv c$.

Any sextic $ax^6 + \cdots$ which represents only cubes, modulo 7, cannot equal -1 for every x and y, not both zero, and hence must take the value $+1$. Hence it can be transformed into $x^6 + \cdots$.

THEOREM. *Every binary sextic which represents only cubes, modulo 7, can be transformed into*

$$(x^2 + y^2)^3 \quad or \quad x^6 + x^3y^3 - y^6.$$

8. Let $\lambda = z^6 + bz^4 + dz^2 + f$, $\mu = cz^3 + ez$. Then $\lambda + \mu$ and $\lambda - \mu$ must be cubes for every z. For $p^n = 13$, $(\lambda \pm \mu)^4 \equiv 1$, whence

$$(17) \quad \lambda \equiv 0,\ \mu^4 \equiv 1;\ \mu \equiv 0,\ \lambda^4 \equiv 1;\ \lambda^2 + \mu^2 \equiv 0, \lambda^4 \equiv \mu^4 \equiv 3 \pmod{13}.$$

Hence for every z,

$$(18) \qquad \mu(\mu^4 - 1)(\mu^4 - 3) \equiv 0 \pmod{13}.$$

If $e \equiv 0$, $\mu = cz^3$, and (18) gives

$$c(c^4 - 1)(c^4 - 3) \equiv 0 \ (\text{mod. } 13).$$

If $c^4 \equiv 1$, then $\mu^4 \equiv 1$ for $z \not\equiv 0$, so that, by (17), $\lambda \equiv 0$ for every $z \not\equiv 0$, contrary to the degree of λ. If $c^4 \equiv 3$, then $\mu^4 \equiv 3$, $\lambda^2 + \mu^2 \equiv 0$ for $z \not\equiv 0$; but the resulting conditions give $b \equiv d \equiv 0$, $c^2 \equiv \pm 3$, contrary to $c^4 \equiv 3$.

For the troublesome case $c = e = 0$, we set

$$\lambda = z^6 + dz^2, \quad \mu = bz^4 + f,$$

so that $S = \lambda + \mu$ is a cube. Replacing z by $5z$, we see that $-\lambda + \mu$ is a cube for every z. Hence one of the three cases (17) must hold. Now $\mu(\mu^4 - 3)$ involves only powers of z^4 and hence, for $z \not\equiv 0$, equals a function of degree 8. Thus at least four values $\not\equiv 0$ of z must give rise to the first case (17). Hence $\lambda = 0$ has four roots $z \not\equiv 0$, so that $d^3 = -1$. Multiplying y by d^{-1}, we may set $d = -1$. Thus $\lambda = z^2(z^4 - 1)$. When $\lambda \not\equiv 0$, we have $z^4 = 3$ or 9, viz.,

$$z^4 = 3, \quad \lambda = 2z^2, \quad \lambda^2 = -1, \quad \lambda^4 = 1;$$

$$z^4 = 9, \quad \lambda = 8z^2, \quad \lambda^2 = 4, \quad \lambda^4 = 3,$$

thus corresponding to the respective cases (17). Thus if $z^4 = 3$, then $\mu = 0$, whence $3b = -f$. For $z^4 = 9$, then $\mu^2 + 4 = 0$, $f^2 = -1$. Hence $S = (z^2 \pm 2)^3$.

Finally, let $e \neq 0$. Since every integer is a fifth power modulo 13, we may multiply y and make $e = 1$. Then (18) holds for $\mu = cz^3 + z$. For $z \neq 0$,

$$\mu^8 - 4\mu^4 + 3 = (8c^7 - 3c^3 + 8c)z^{10} + (2c^6 + 2c^2 + 1)z^8$$
$$+ (4c^5 - 3c)z^6 + (5c^4 - 4)z^4 + 4c^3z^2 + c^8 - 4c^4 + 2c^3 + 3.$$

In its product by $cz^2 + 1$, the coefficients of z^4 and z^6 are

$$9c^4 - 4, \quad 9c^5 - 7c,$$

which do not both vanish modulo 13.

Hence the sextics which represent only cubes modulo 13 have been reduced to the product of $(x^2 \neq 2y^2)^3$ by a cube l^3. But $l(x^2 \pm 2y^2)$ can be transformed into $x^2 + 2y^2$.

THEOREM. *Every sextic which represents only cubes modulo 13 is a perfect cube which may be transformed into* $(x^2 + 2y^2)^3$.

9. For the next two values 19 and 25 of $p^n = 3k + 1$, I find that any binary sextic S which represents only cubes equals the cube of a quadratic form. For $p^n = 5^2$, this result follows readily by requiring that S^2 shall equal its fifth power; by the coefficients of z^{21}, z^{16}, z^{11}, z^4, we get $S = (z^2 + 2b)^3$.

For $p^n = 11^2$, S^4 must equal its 11th power. By the coefficients of z^{111}, z^{100}, z^{89}, z^{78}, we get $S = (z^2 + 4b)^3$.

It seems probable that, for $p^n = 3k + 1 > 7$, every binary sextic which represents only cubes is formally a perfect cube. Although certain general considerations point to the validity of this conjecture, I have not effected a complete proof.

10. A sextic on three or more variables cannot represent exclusively cubes in a field of order $p^n = 3k + 1$ for which the binary sextics representing only cubes are formally perfect cubes. It suffices to prove the theorem for ternary sextics T. For* $p > 2$, we may delete the terms x^5y, x^5z, y^5z. Then the terms free of z are $(x^2 - vy^2)^3$, where v is a not-square. When z is multiplied by a suitable constant, the terms free of y are $(x^2 - vz^2)^3$. Then the terms free of x are $-v^3(y^2 + \epsilon z^2)^3$, where $\epsilon^3 = 1$. Since ϵ equals the square of ϵ^2, and $y^2 + \epsilon z^2$ is irreducible, -1 must be a not-square. Hence we may set $v = -1$. Hence

$$T = x^6 + y^6 + z^6 \tag{19}$$
$$+ 3(x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + \epsilon y^4z^2 + \epsilon^2y^2z^4) + xyz\phi,$$

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* By a different argument, I have proved that there is no tenary sextic in the $GF[2^{2m}]$ which represents only cubes.

$$
(20) \qquad \phi = Ax^3 + By^3 + Cz^3 + Dx^2y + Ex^2z + Fxy^2 + Gxz^2 + Ky^2z + Lyz^2 + Mxyz.
$$

Set $x = \lambda y$. Then, for λ arbitrary in the field,

$$
f = (\lambda^2 + 1)^3y^6 + ry^5z + sy^4z^2 + (G\lambda^2 + L\lambda)y^3z^3 + ty^2z^4 + z^6
$$

must be formally a perfect cube in x, y. Here

$$
\begin{aligned}
r &= A\lambda^4 + D\lambda^3 + F\lambda^2 + B\lambda, \\
s &= 3\lambda^4 + E\lambda^3 + M\lambda^2 + K\lambda + 3\varepsilon, \\
t &= 3\lambda^2 + C\lambda + 3\epsilon^2.
\end{aligned}
$$

Hence $f = [(\lambda^2 + 1)y^2 + \omega z^2]^3$, where $\omega^3 = 1$. Thus

$$
r = 0, \quad s = 3\omega(\lambda^2 + 1)^2, \quad t = 3\omega^2(\lambda^2 + 1), \quad G\lambda^2 + L\lambda = 0
$$

for every λ in the given field of order > 4. Hence

$$
A = B = C = D = E = F = G = K = L = 0, \quad M = 6, \quad \omega = \epsilon = 1, \quad T = (x^2 + y^2 + z^2)^3.
$$

By § 2, T vanishes for x, y, z, not all zero, in the field.

11. It remains to investigate the ternary sextics T which represent only cubes in the $GF[7]$. The terms free of x in T will be denoted by B_{yz}, those free of y by B_{xz}, etc. By § 7, each B is a perfect cube or a product π of three quadratic forms. Suppose that one of the B's, say B_{xz}, is of the type π. Replacing x by $x + ry + sz$, we may delete the terms x^5y and x^5z; then B_{xz} is still of the type π, and B_{xy} is of its original type. Hence we may set

$$
B_{xz} = z^6 + x^3z^3 - z^6.
$$

Let T become T' when y is replaced by $y + \rho z$. The coefficient c of z^6 in B'_{xz} is the value of B_{yz} for $y = \rho$, $z = 1$, and hence is a sextic in ρ with a non-vanishing coefficient for ρ^6; thus $c \neq -1$ for some value of ρ. Since B'_{xy} is not of type π, it is a perfect cube. Also, $B'_{xy} = B_{xy}$.

Thus one of the B's may be assumed to be a perfect cube. With B_{xy} a cube, the same argument shows that we may make also B_{xz} a perfect cube, and that we may set

$$
B_{xz} = (x^2 + z^2)^3, \quad B_{xy} = (x^2 + y^2)^3.
$$

Employing the abbreviation ϕ, given by (20), we have

$$T=x^6+y^6+z^6+3x^4y^2+3x^4z^2+3x^2y^4+3x^2z^4+\sum_{i=1}^{5} c_i y^{6-i}z^i+xyz\phi.$$

Set $y=\lambda z$. Then T becomes

$$t=x^6+r_2x^4z^2+r_3x^3z^3+\cdot+r_6z^6,$$

$$r_2=3\lambda^2+A\lambda+3,\ r_3=D\lambda^2+E\lambda,\ r_4=3\lambda^4+F\lambda^3+M\lambda^2+G\lambda+3,$$

$$r_5=B\lambda^4+K\lambda^3+L\lambda^2+C\lambda,\quad r_6=\lambda^6+c_1\lambda^5+\cdots+c_5\lambda+1.$$

Now $\tau=x^6\pm x^3z^3-z^6$, viz., is of type π, if and only if

$$(21)\qquad r_2\equiv r_4\equiv r_5\equiv 0,\quad r_3^2\equiv 1,\quad r_6\equiv -1\qquad (\text{mod } 7);$$

while τ is a perfect cube if and only if

$$(22)\qquad r_3\equiv r_5\equiv 0,\quad r_4\equiv 5r_2^2,\quad r_6\equiv 6r_2^3\qquad (\text{mod } 7).$$

Since $r_2r_3\equiv 0$ for every λ, $D\equiv E\equiv 0$. Hence (21) is excluded, so that (22) must hold for every λ. We may therefore remove the term y^5z from T and proceed as in § 10. Or we may proceed with (22) and show that

$$T\equiv(x^2+y^2+z^2-2Ayz)^3.$$

12. The theorem that there exists no sextic on three or more variables which represents only cubes in a field of order $p^n=3k+1$ has now been established for $p^n<31$, $p^n=2^n$, and $p^n=11^2$. Its truth for all values of p^n has been proved, subject to the validity of the conjectured theorem of § 9 on binary sextics.

THE UNIVERSITY OF CHICAGO,
October, 1908.

NOTE ON LÜROTH'S TYPE OF PLANE QUARTIC CURVES.

BY PROFESSOR H. S. WHITE AND MISS KATE G. MILLER.

(Read before the American Mathematical Society, September 6, 1907.)

ONE of the stock examples of the fallacy of constant counting is the equation of a plane quartic, whose fourteen constants equal in number those apparent in the sum of five fourth powers of linear expressions

$$x_1^4+x_2^4+x_3^4+x_4^4+x_5^4.$$

Clebsch showed why this is not a normal form for all ternary quartics; it has by virtue of its form an infinite system of linear transformations into itself, and so contains actually but thirteen constants. The curve has an apolar, *i. e.*, a triply apolar conic, and the sides of its polar quinquilaterals form a simple quintic involution of tangents to this conic. This special type of quartic is called the Clebschian.

Lüroth's quartic, closely related to this, is an equally alluring source of fallacy. It contains inscribed complete quinquilaterals, — all ten vertices lying on the curve. At first sight one would say that every quartic might be of this type. For we may take three variable points, A, B, C on the curve, join them by straight lines, and so have a triangle with three degrees of freedom. Each side cuts the curve in two more points, say L_1 and L_2 on BC, M_1 and M_2 on CA, and N_1, N_2 on the third side. One condition is imposed on the variable points by requiring three points L_1, M_1, N_1, to be collinear; let their line meet the quartic again in P. Finally we should expect two more conditions to be imposed by requiring the four points L_2, M_2, N_2, and P to be collinear; three conditions in all upon the three variable points A, B, and C. If so, the construction might be possible for a finite number of positions, and for each position the five lines of the diagram would constitute an inscribed complete quinquilateral. Unfortunately for the argument, one such inscribed quinquilateral in a quartic curve proves the existence of an infinite number of others instead of a finite number, hence our hasty inference that this type is the most general quartic is nugatory. Lüroth's quartic and the Clebschian are related by the fact that all the inscribed quinquilaterals of the one are the polar quinquilaterals of the other.

Lüroth's quartic is evidently a projective covariant curve of the Clebschian, and has been studied as such. If a certain invariant B (the catalecticant) of the ternary quartic c_x^4 or c vanishes, then c is of the Clebsch type, and its covariant S gives the equation $S = 0$ of the corresponding Lüroth quartic

$$S = (cc'c'')(cc'c''')(cc''c''')(c'c''c''')c_x c'_x c''_x c'''_x = 0.$$

The direct and unique connection between the two is thus completely exhibited when the Clebschian is given as the starting point. From it both the Lürothian and the related conic are found as rational covariants,* and the condition which must be satisfied is expressed by a rational invariant equation. But suppose the Lürothian equation to be given, $l_x^4=0$. Could we then find rationally in its coefficients the Clebschian, the conic, and the characteristic invariant condition for the Lürothian type? This has not been announced, yet apparently it should be possible. We wish here to outline a possible answer, although the result is not so simple as to be entirely satisfactory.

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* See statements and references by Coble in *Transactions Amer. Math. Society*, vol. 4 (1903), pp. 65–85.

Let the Lürothian, Clebschian, and conic be represented by the equations

$$l_x^4=0, \quad c_x^4=0, \quad \text{and} \quad u_a^2=0,$$

respectively. These are connected by a relation that seems not to have been employed hitherto. It is well known that tangents to the conic are cut by the two quartics in apolar quartettes of points. But it is further true that *these tangents are the only lines whose intersections with the quartics are mutually apolar*. The envelope of such lines is of the fourth class,

$$(clu)^4=0,$$

and if the conic forms part of this envelope, a second conic must form the complementary part. That this complementary part is the same conic repeated can be verified easily from the normal forms of the equations

$$l_x^4=\sum_1^5 (x_2x_3x_4x_5) \quad \text{and} \quad c_x^4=\sum_1^5 x_1^4.$$

We have therefore the formal equation, identical in the u's,

$$(1) \qquad (lcu)^4 \equiv (u_a^2)^2.$$

The triple apolarity of conic and Clebschian is expressed by the identity in the variables x_1, x_2, x_3,

$$(2) \qquad c_a^2c_x^2 \equiv 0.$$

The identity (1) yields 15 equations linear in the quantities $c_{ijk}(=c_1^ic_2^jc_3^k)$; from these we find the Clebschian explicitly as a fifteen-rowed matrix bordered with one column of the second degree in the coefficients of u_a^2, and one row quartic in variables x_1, x_2, x_3. Call the determinants $|L|$, and the bordered matrix

$$\begin{vmatrix} L & (a)^2 \\ (x)^4 & 0 \end{vmatrix},$$

and we write

$$(3) \qquad |L| \cdot c_x^4 \equiv \begin{vmatrix} L & (\alpha)^2 \\ (x)^4 & 0 \end{vmatrix}.$$

Hence, applying condition (2), we find what may be symbolized thus

$$(4) \qquad \begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, x^2) & 0 \end{vmatrix} \equiv 0 \text{ for all values of the } x_1, x_2, x_3.$$

This contains six cubic equations for the six coefficients of the conic. *If these can be satisfied simultaneously, the quartic* l_x^4 *is of Lüroth's type; equation (3) gives the Clebschian in terms of the Lürothian and the conic; and the common solution*

$$(\alpha_{11}, \alpha_{12}, \cdots, \alpha_{33})$$

of the six equations (4) is the conic envelope.

We may state the relation of things more concisely by using the terminology of space of five dimensions, treating the coefficients α_{ik} of the conic as homogeneous coordinates in such a space. Replace the powers and products of x_1, x_2, x_3 in equation (4) by a second set of coordinates β_{ik} cogredient with the α_{ik}; then equation (4) is that of the first (cubic) polar of a point β in the five-space, with respect to a quartic locus or four-spread in that space, whose equation may be denoted by

$$(5) \qquad \begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, \alpha) & 0 \end{vmatrix} = 0,$$

since

$$\begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, x^2) & 0 \end{vmatrix}_{x_i x_k = \beta_{ik}} \equiv \tfrac{1}{2} \Sigma \left(\beta_{ik} \frac{\partial}{\partial \alpha_{ik}} \right) \cdot \begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, \alpha) & 0 \end{vmatrix}.$$

This first polar equated to zero, identically in the β_{ik}, is the requirement that the quartic four-spread (5) shall have an ordinary double point or node, viz., at the point (α). Therefore, *the problem of finding the invariant condition satisfied by the Lürothian is equivalent to the problem of finding the discriminant of a certain quartic (5) homogeneous in six variables. If that discriminant vanishes, the determination of the covariant conic* α *is effected by determining the coordinates of the node of that quartic four-spread; and from this conic and the Lürothian the Clebschian* c_x^4 *is then found uniquely by equation (3).*

That this solution shall not be illusory, of course the determinant $|L|$ must not vanish. For a particular quartic, one containing only even powers of x_1, x_2, x_3, this test is made very easily; and it is of interest to record that then this determinant $|L|$ factors into two rational expressions of degrees 6 and 9 respectively, neither of them containing Salmon's invariant B as a factor. This shows that $|L|$ is not zero for a quartic L of the Clebschian type. But also for the Lüroth type of quartic it does not vanish. This we have proved with some labor by taking l_x^4 in the canonical form

$$l_x^4 = Ax_2x_3x_4x_5 + Bx_1x_3x_4x_5 + \cdots + Ex_1x_2x_3x_4,$$

with the two linear relations

$$x_4 = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3, \quad x_5 = \beta_1x_1 + \beta_2x_2 + \beta_3x_3,$$

and computing a sample term of $|L|$. We selected the one containing the quantities C and β_3 to the highest possible powers, namely the term in

$$A^2B^2C^{11}\alpha_1^7\alpha_2^5\alpha_3^3\beta_2^2\beta_3^{13}.$$

Such a term can occur in only 21 terms of the expanded determinant $|L|$, and carefully repeated computation gives its coefficient as -8. While we should prefer some simpler verification of this point, yet we regard this trial as sufficient.

Owing to the high degree of the discriminant of a senary quartic, it seems at least likely that for a quartic of the particular type $\left|\begin{smallmatrix} L & (\alpha, \alpha) \\ (\alpha, \alpha) & 0 \end{smallmatrix}\right|$ the discriminant might prove to be an exact power of some invariant of lower order in the given coefficients l_{ik}. Such a reduction we have proved to exist, so far, only in certain analogous forms in fewer variables.

When the Lürothian is regarded as derived from a quintic involution among the tangents of a conic, F. Meyer has shown precisely how to determine the (4, 4) correspondence among those tangents that represents the Clebschian. This mode of determination he extends to forms of any even order. An involutional sextic curve is always uniquely associated with another sextic which is apolar to the septilaterals inscribed in the former. The relation between such a pair of sextics (octics, etc.) in ternary variables can be expressed by an obvious generalization of the identities (3) and (4). For example if $l_x^6 = 0$ and $c_x^6 = 0$ are two such sextics, and $u_a^2 = 0$ the conic carrying the related involution, then the envelope

$$(clu)^6 = 0$$

is the triply counted conic: $(clu)^6 \equiv \rho \cdot (u_a^2)^3$. So for the similarly related pairs of higher order.

VASSAR COLLEGE.

CANTOR'S HISTORY OF MATHEMATICS.

Vorlesungen über Geschichte der Mathematik. By MORITZ CANTOR. Vierter Band. Von 1759–1799. Leipzig, Teubner, 1908. 8vo. vi + 1113 pp. 32 Marks.

IT is a long period that Moritz Cantor looks back upon to the time when he made his first noteworthy contribution to the history of mathematics. He was little more than a boy when he published his Inaugural-Dissertation "Ueber ein weniger gebräuchliches Coördinaten-System" in 1851,—only twenty-two years old; and it was only five years later that he entered upon his life work as historian of mathematics, by the publication of his paper "Ueber die Einführung unserer gegenwärtigen Ziffern in Europa." He was thirty-four when his first noteworthy treatise on the history of mathematics appeared, the "Mathematische Beiträge zum Kulturleben der Völker" (Halle, 1863). And next August he will be eighty years old, with over a dozen books to his credit, with hundreds of memoirs, with a notable record as editor of the *Zeitschrift für Mathematik und Physik* and the *Abhandlungen zur Geschichte der Mathematik*, and with academic honors and governmental recognition commensurate with the great work that he has accomplished.

For a man of nearly eighty to undertake a work of the magnitude of this fourth volume of the Geschichte would seem foolhardy, were he not such a man as Professor Cantor. When the third volume was completed in 1898 it seemed proper that it should be called the last, as was the case, and under all the circumstances any man might have been well content to say his *nunc dimittis*. That Professor Cantor was not thus content is a cause of gratification to all who are interested in the history of mathematics, and a worthy lesson to all who feel that three-score years mark the bounds of man's working life.

The present volume is not the work of Cantor himself in the same way as the earlier part of the history, although it was planned and edited by him and contains one chapter from his own pen. He has brought to his assistance several of the best men in Germany and in other countries, each an expert in the line upon which he has written. The following are the chapters and authors: Histories of mathematics, by S. Günther; Arithmetic, algebra, and theory of numbers, by F. Cajori; The combination theory, doctrine of probabilities, series, and imaginaries, by E. Netto; Elementary geometry, by V. Bobynin; Trigonometry, polygonometry, and tables, by A. von Braunmühl; Analytic geometry of the plane and of space, by V. Kommerell; Perspective and descriptive geometry, by G. Loria; Infinitesimal calculus, by G. Vivanti; Total and partial differential equations, sums and differences, and the calculus of variations, by C. R. Wallner; Survey of the period from 1758 to 1799, by M. Cantor.

Professor Günther's article relates to the histories of mathematics, including monographs and editions of the classics, published during the period in question. The chapter is particularly valuable for the reason that we have no worthy bibliography of the subject in any period preceding the founding of *Bibliotheca Mathematica* and the *Jahrbuch über die Fortschritte der Mathematik*, and this particular period is practically the one in which serious historical investigation begins. A rather careful examination of this chapter fails to show the omission of any names of importance, and the chapter will stand as a worthy bibliography of the period. One or two minor misprints have been noticed, but they are not worth mention at this time.

It is a matter of gratification to Americans that Professor Cajori should appear as one of the collaborators in this work. To him was assigned the subject of arithmetic, theory of equations, and theory of numbers. The period has not been carefully studied before, and it is apparent even on first reading that Professor Cajori's best work has been done here. He has gone to a sufficient number of original sources to trace the development at first hand, and his secondary authorities are standards. Not the least valuable is the brief but comprehensive sketch of early American textbooks. Some points are open to criticism, and naturally so when secondary authorities are taken. For example, Trenchant (1566) commonly used miliar for 10^9, and miliar de miliars for 10^{12}. As to the date of Ruffini's Teoria generale delle equazioni, upon which Professor Cajori raises a question, it is not improbable that the authorities giving 1798 (and the excellent Elogio by Giuseppe Bianchi might well have been included) are quite as correct as those giving 1799. In my own copy the title pages of both volumes originally read MDCCXCVIII, but in both cases an additional I has been added, perhaps by a type held in the hand. It is probable that some questions may arise with respect to a few of the other dates, particularly of memoirs, for so much material has been used that slight errors are to be expected.

Netto's contribution begins with the so-called kombinatorische Schule, founded by Hindenburg whose first publication on the subject was made at the age of thirty-nine, although Æpinus, van Schwinden, and Euler had already made some use of the idea in connection with the binomial series, the last named having suggested the common German symbolism for the coefficients and having stated two well-known problems in the theory. In the doctrine of probabilities the author shows our great indebtedness to D'Alembert, who, building upon the work of Jakob Bernoulli, contributed much more to the theory than has commonly been supposed. The work of Laplace has generally been appreciated, probably because of the extensive use of least squares, and possibly because of the interest immediately aroused by his study of vital statistics, and Professor Netto has given an excellent resumé of this phase of Laplace's work. In the study of series the beginning is made with the work of Landen. This is followed by the contributions of Euler, Daniel Bernoulli, Pfaff, and other continental writers, and also of the too generally neglected English scholars, Vince, Waring, and Hellins. In the subject of imaginaries Professor Netto also gives much more credit to D'Alembert than has generally been the case. He shows that Euler first used i for $\sqrt{-1}$, in his Institutiones calculi integralis appearing the words, "formulam $\sqrt{-1}$ littera i in posteriorem designabo." As to the graphic representation of complex numbers he gives the credit, of course, to Wessel.

Professor Bobynin begins his chapter on elementary geometry by a discussion of textbooks, showing very effectively by table the great advance of Germany and France, in this phase of education, over the other European countries, and especially over England. Here the influence of D'Alembert again stands out prominently, and Bobynin's appreciation following immediately that of Netto suggests that in spite of the praise that

D'Alembert has generally received he is entitled to still more recognition. It is, however, to Legendre that the author assigns the greatest honor, and with good reason, and the essential features of his revision of the Euclid sequence are carefully set forth. The second topic in this chapter is on practical geometry or surveying, and here the noteworthy advance naturally centers around the establishing of the basal unit of the metric system. We are already in possession of an excellent treatise on this subject in the work of Bigourdan, but Professor Bobynin has approached it more definitely from the geodetic standpoint. The third topic relates to specific propositions, and particularly to Lexell's contributions. The fourth topic considers the theory of parallels, necessarily closing before the modern Lobachevsky-Bolyai developments.

The contribution of the late Professor von Braunmühl is along the line of his great life work, the history of trigonometry. It begins with the period of Euler and his contemporaries, the same as is covered, often more fully, in the author's history, chapters 4 and 5. The date of Blake's birth, on the second page, is wrong, by the way; it should be 1708 instead of 1718. The period is one of peculiar interest, covering a good part of Euler's most prolific years, but the treatment is necessarily more condensed than in von Braunmühl's larger work. The second topic relates to efforts to improve the teaching of trigonometry, especially as to its foundation principles. The third part relates to tetragonometry, polygonometry, and polyhedrometry, and the fourth part to tables, cyclometry, and trigonometric series. In several places the latter part of von Braunmühl's contribution is more complete than his history.

Professor Kommerell's chapter on analytic geometry of the plane and of space covers a field that has been less worked than that of the preceding chapter or two. He begins with some general notes on conics and then takes up higher plane curves, proceeding from Waring's Proprietates algebraicarum curvarum, of which he gives a careful resumé. The period was one of great activity in this line of work, and some fifty pages are assigned to the subject. Space curves and surfaces are next discussed, beginning again with Waring, this part of the treatment also covering about fifty pages.

Professor Loria's article is divided into four topics. The first treats of perspective, but unlike the other contributors the author has found it necessary to begin with the middle ages on account of omissions in the earlier volumes of the history. He brings out in a clear if not new light the indebtedness of mathematics to the graphic arts in the fifteenth and sixteenth centuries, and makes further mention of the work of Desargues in the seventeenth century. His second topic is The golden period of theoretic perspective, in which he begins with the work of s'Gravesande and properly pays high tribute to the genius of Lambert. The third topic touches briefly upon the immediate predecessors of Monge, and the fourth devotes some fifteen pages to the great founder of descriptive geometry.

The treatment of the infinitesimal calculus by Professor Vivanti opens with a consideration of the foundations of the subject, carrying the work on from where it was interrupted in volume III. The second topic deals with the textbooks of the period. The third is subdivided, the first part treating of processes of differentiation, in which the work of Pfaff is most prominent, and the second treating of integration, a subject particularly fertile of results in this period. The fourth topic relates to definite integrals, beginning and necessarily chiefly concerning itself with the contributions of Euler. The fifth topic relates to the analytic applications of the infinitesimal calculus, beginning with the English contributors to the theory of maxima and minima, and including the development of indeterminate forms and series. Professor Vivanti's contribution closes with an exhaustive discussion of transcendents and elliptic integrals, some eighty pages in all, beginning with Vandermonde's treatment of the theory of factorials, but chiefly devoted to the theory of elliptic integrals. These functions, while first suggested somewhat earlier, had their greatest development in this period, and chiefly at the hands of Euler. While the subject has already been well treated by Enneper and by Bellocchi, the author has examined at first hand so much source material that his own work must be used to supplement that of his two predecessors.

Professor C. R. Wallner's contribution to the history of total and partial differential equations, differences and sums, and the theory of variations, is one of the most exhaustive in the work. While the theory of differential equations had already been established before 1750, their solution at the outset had been looked upon as means to an end rather than an object in itself. It was during the latter half of the eighteenth century, the period covered by this volume, that the theory came to its complete independence, and chiefly through the influence of Euler, D'Alembert, Danie Bernoulli, Lagrange, Laplace, and Legendre.

Professor Cantor's own contribution is largely bibliographical, consisting of a list of the most important works mentioned by his collaborators. He makes the same assignment of date to Ruffini's first work on equations of the fifth degree as made by Professor Cajori. It is evident that the printing of the work began in 1798 and was completed in 1799.

It is too early to enter into critical details of such a work. That numerous errors will be found is certain, as witness the partial list given by Müller, and the fact that the *Bibliotheca Mathematica* gives every month a list of corrections extending back to the first edition of the first volume, published nearly thirty years ago. On the other hand it will be a long time before any one will attempt to treat so exhaustively this remarkable period in which the genius of Euler, D'Alembert, Lagrange, Laplace, and Legendre showed at its best, and in which Gauss was beginning the labors that placed his name among the leaders of modern times. DAVID EUGENE SMITH.

SHORTER NOTICES.

High School Algebra. By H. E. SLAUGHT and N. J. LENNES. *Elementary Course,* 1907, vii + 297 pp., $1.00; *Advanced Course,* 1908, vii + 194 pp. $0.65. Boston, Allyn and Bacon.

THE common weakness of our college students in elementary algebra shows a great need for improvement in the teaching of this subject. The appearance of these admirable texts, constructed after a new model, marks a distinct advance in the teaching of algebra in our high schools. Recent discussions have shown that it is quite generally agreed that high school algebra should be divided into a first course of elementary algebra during the first year, and a second course of review and advanced algebra during one half of the third or fourth year, preceded by one year of plane geometry. The authors have met this demand of teachers by dividing their text into two parts, and in doing so have succeeded in presenting the subject of high school algebra in the most concrete and teachable form we have yet seen.

As stated in the preface, "the main purpose of the Elementary Course is the solution of problems rather than the construction of a purely theoretical doctrine as an end in itself." Throughout the first course the method is inductive. The interests, ability, and needs of the first year high school students seem to demand this selection of the inductive method. In writing elementary texts, mature mathematicians too often let their delight in the subject as an elegant deductive system obscure the ability of the student at this particular stage in his development. The teachableness of this book is greatly increased by the fact that the authors never lose sight of the standpoint of the *pupil.*

The illustrative problem is used throughout the book to show the need of new processes and to secure real insight into the principles involved. This enables the authors to build upon the arithmetic knowledge and experience of the pupils, to proceed from the known to the unknown, and to make the work more concrete. Illustrative problems are made to take the place of long abstract descriptions and explanations. New notions and processes thus first made evident by illustrative exercises are always followed up by a careful and precise statement of the *principle* involved. By this method eighteen principles used in the Elementary Course are introduced. These eighteen principles stand out prominently as a body of facts which each student must know and be able to use. It is somewhat of a surprise that the authors have been able to reduce the subject of elementary algebra to such a small number of fundamental principles.

New ideas are introduced one at a time and only when needed in the solution of problems. This, with a desire to secure a maximum of concrete matter in the early parts of the book, has led to a slightly new order of topics. The most noticeable change from the conventional order is in the position of factoring, long division, and literal fractions. Factoring follows simultaneous linear equations, where first needed as an introduction to the quadratic. Long division comes quite late in the course because it is first needed in the square root process required in the general solution of the quadratic. Literal fractions come in the last chapter because the knowledge of arithmetic fractions and the use of simple operations are sufficient for earlier problems. This new order of topics is justified by the gain in more early concrete work and in placing the complicated work later in the course. Of course such an order of topics requires greater care in the grading of problems, and this has been exercised.

Since the chief purpose is the solution of problems, the equation as an instrument for this purpose is early introduced and occupies a prominent place throughout the book. The analogy drawn between a balance and an equation is good. Equations are solved for many different letters, which is excellent preparation for later work in physics. The real test for a solution, namely that it satisfies the equation, is properly emphasized by many exercises in checking. The form of arrangement in the illustrative problems requires a justification of each operation by an explicit reference to some one of the eighteen fundamental principles. This tends to overcome the common fault of blind juggling of symbols and lays an excellent foundation for the deductive proofs of the Advanced Course. On nearly every page appear evidences of a purpose to develop ability to translate English into the forms of algebra; many parallel statements of the same fact in English and algebra are found; formulas are restated in words; numerous exercises in translation are given.

Negative numbers are well introduced and treated by means of concrete examples; emphasis is placed on the interpretation of the sign of the result of a problem. The term "less than zero" which confuses beginners is avoided. The term "absolute value" is introduced in this connection.

The justification for the introduction of graphs into algebra is their usefulness in illustrating the principles involved in simultaneous equations. The authors introduce graphs just *before* considering linear equations and then *use* them in their treatment of simultaneous equations throughout both books. The usefulness of graphs is made so evident that they become an integral part of the subject, instead of seeming to be an extraneous topic tacked on to algebra.

The emphasis placed upon elimination by substitution should be noted. Real insight into what is happening comes through the process of substitution. Elimination by addition and subtraction is given second place. Elimination by comparison is omitted in the Elementary Course but is found in the Advanced Course.

Factoring is introduced on page 172 — about the middle of the course. Complicated cases are omitted in the first course. The treatment of the type $ax^2 + bx + c$ is especially good. In each set of exercises for factoring there are some expressions which are not factorable by the case under consideration; this has the advantage of requiring the pupil to study the *form* of each expression.

The treatment of the quadratic illustrates the authors' use of the spiral method. The quadratic first appears on page 197 as an application of factoring; on page 240 as an application of square root; quadratic and linear simultaneous equations come on page 246; while the complete theory of the quadratic comes in the Advanced Course. This spiral method has the advantage of going from the simple to the complex and gives a good review of a topic by repeated considerations, each time in a more complicated form.

Literal fractions appear in the last chapter. It is surprising how many problems can be solved without the use of literal fractions. In the treatment of complex fractions the shorter method of multiplying both terms of the complex fraction by the least common multiple of all the minor denominators is used. This method is too frequently neglected.

Numerous definitions appear in the Elementary Course, but a tendency to avoid or postpone the introduction of technical terms is noted. The authors seem to feel that by repeating the description of technical terms, such as *transpose* or *equivalent equations*, each time they are needed a greater insight into real processes is obtained.

The reviewer was impressed by the number of concrete problems; in the first 150 pages there are about 65 pages of concrete problems and 25 pages of abstract exercises. The problems contain interesting information and are of such a nature as constantly to raise the question whether the result is reasonable. The desirable habit of appealing to one's common sense to justify results is thus strengthened. The inductive principle is even carried into the problems. General or literal problems which result in formulas are approached through a series of numerical problems. The close connection of algebra and arithmetic is brought out by many problems in Arabic numbers; nearly every principle is illustrated by Arabic numbers.

The Advanced Course presupposes a knowledge of the principles of elementary algebra, considerable training in deductive reasoning obtained in the preceding year of plane geometry, and greater maturity in the pupils; hence its point of view is entirely different from that of the Elementary Course. Its method is deductive, and instead of principles to be illustrated the algebraic facts are put in the form of theorems to be proved. This new point of view gives a fine review of the Elementary Course without degenerating into an uninteresting rehash of old work.

The first chapter is given to a careful statement of the axioms used. The nine axioms cover the assumptions made concerning the uniqueness of addition, subtraction, multiplication, and division and their commutative, associative, and distributive laws. The exclusion of division by zero is properly emphasized here; throughout both books many notes call attention to this exclusion of division by zero.

A good treatment of equivalent equations is found in Chapter III. A complete treatment of factoring, including the factor theorem, appears in Chapter V. The graph is continually used in simultaneous equations. The theory of exponents and radicals is completed in this course. The discussion of formulas as a preparation for later work in physics is emphasized. Radicals are well treated and followed by a good collection of problems involving radicals.

The topics covered in the two books fully meet the present college requirements with additional chapters on variation, logarithms, arithmetic and geometric progressions, and the binomial theorem for positive integral exponents.

The following definition of irrational number seems unnecessarily complicated and incomplete: "If a number is not the kth power of an integer or a fraction, but if its kth root can be approximated by means of integers and fractions to any specified degree of accuracy, then such a kth root is called an irrational number" (page 70). The incompleteness is willful, for the authors later remark, "There are other kinds of irrational numbers besides those here defined." No *explicit* definition of rational numbers was found. There seems to be a slight inconsistency between this discussion of numbers on page 70 and that of roots on page 11.

We regret to see no use of the symbol i for the complex unit, and the omission of the term *discriminant* in the discussion of the quadratic. An index of definitions would be a useful addition to both books. There is no appendix. The authors have had the courage to omit entirely topics which they believe have no place in a high school algebra, such as simultaneous equations of more than three unknowns (a single example in four unknowns is given) inequalities, complicated complex fractions.

Judged as a whole, these books are the best texts on high school algebra we have seen, and their use will produce stronger students than we are now receiving from the high schools.

ERNEST B. LYTLE.

Einführung in die Hauptgesetze der zeichnerischen Darstellungsmethoden. By Professor ARTUR SCHOENFLIES. B. G. Teubner, Leipzig and Berlin, 1908. iv + 92 pages. 98 figures.

DURING the last few years the need for more systematic and comprehensive instruction in geometric drawing has become much more keenly felt among school and college teachers. While the courses in descriptive geometry for technical students are frequently sufficiently extensive to claim nearly half the time of the student for a year or more, the corresponding training for prospective teachers is usually very inadequate. Among the attempts to supply this need, numerous recent publications of the firm of B. G. Teubner are worthy of careful consideration. The question may be fairly put, whether the emphasis should be placed on the technical details or on the geometric principles underlying them. The little book of Schütte* is of the first kind; that of Müller and Presler† and also that of Schüssler‡ are much more comprehensive and combine considerable instruction in geometry with the explanations of the constructions. But the book under review puts most of the emphasis on the geometric principles. It is meant for much more mature students than the other books mentioned. The aim is frequently not so much the acquisition of sufficient knowledge or skill to introduce all the geometric details into a figure, but rather the power of making a good illustration by means of a few lines drawn free hand.

After a good discussion of the principles of plane perspective the theorem that any two projective pencils have one right angle formed by corresponding lines is proved in great detail. This theorem is used later. The treatment of the methods of descriptive geometry is brief, but clear. A reader of fair ability and much patience could read these parts profitably without a teacher, though too few exercises are given to prove his interpretations or measure his skill. Now follow in rapid review the outlines of four important methods, though much too brief to be of maximum service. The treatment of axonometry is confined almost entirely to the proof, elegant in itself, of Polke's theorem that any plane quadrilateral is similar to some plane (parallel) projection of a given tetrahedron. Nine interesting applications are hastily sketched. The determination of apparent contour is the most interesting part of the book; the applications to the anchor ring and hyperboloid of revolution could hardly be followed by a reader to whom these figures are new. Stereographic projection is theoretically discussed, the invariant property of circles and angles being established. The depiction of the edges of a cube on a plane perpendicular to a diagonal is the only illustration. Finally, a few pages are devoted to relief perspective, generalizing for three dimensions what was done earlier in the book for two. A number of instructive notes are added, more fully explaining various points in the text.

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* See review in BULLETIN, vol. 14, p. 294.

† See BULLETIN, vol. 10, p. 207.

‡ Reviewed in BULLETIN, vol. 12, p. 361.

The only typographical errors that would cause confusion are on page 23, where regions II, III should be interchanged, and on pages 32, 33, where the letters F_0, F_2 in the text refer to P_0, P_2 respectively in the figures.

VIRGIL SNYDER.

La Géométrie Analytique Générale. Par H. LAURENT. Paris, A. Hermann, 1906. vii + 151 pp.

MANY text-books, chiefly on analysis and algebra, have been written by M. Laurent during the last twenty-five years. The present little volume, with its far-reaching title, suggests that the author is turning his attention, for the moment, to the field of geometry. One soon finds that the underlying thought is that geometry is merely a branch of the theory of numbers. The author has certainly succeeded admirably in emphasizing the fact that geometry may be considered as a purely abstract subject capable of treatment by analytical methods without reference to space perceptions. However, the work is not, as a whole, just what one would desire as a brief first book on the subject of "General analytical geometry." For instance, one feels that much of the strength of the book has been lost by spending an undue amount of time on the one topic of orthogonal substitutions and by delaying the definition of a group too long.

After a three page introduction on trigonometry (where the ratios are defined analytically), the reader is plunged at once into the main topic of the book—orthogonal substitutions. In the conclusion half a dozen pages are devoted to some general remarks about groups of substitutions. A definition of a group is given, and the statement is made that the orthogonal substitutions form a group. In the next paragraph the author says that all possible geometries are, fundamentally, the study of groups of substitutions and their invariants. Should this most important fact be relegated to the conclusion and there passed over lightly? May not the student regard it as a mere afterthought? Is it one of the conclusions that have necessarily arisen in his mind during the study of this book? Indeed, one fears that he would not have sufficient breadth of view to appreciate the meaning of the sentence.

Another criticism that must be made is in regard to the absence of any trace of a bibliography. It may be unnecessary to give references in secondary school books (though one can certainly raise the question even there); but it does seem most unwise to omit from a text of this nature all reference to the literature of the subject. E. B. COWLEY.

Tafeln unbestimmter Integrale. Von G. PETIT BOIS, Bergingenieur in Lüttich. Leipzig, Teubner, 1906. 4to. xii + 154 pp.

THE book contains 2,500 or more indefinite integrals, unnumbered. This is five times as many as occur in B. O. Peirce's Short Table of Integrals, but the latter would be, in general, the more useful book on account of its containing also definite integrals and many auxiliary formulas. In the three-page index to the Tafeln are given the 110 groups into which the formulas are divided. About three-fourths of the integrals are of algebraic functions. As an introduction, there are 49 transformation formulas. The books most used in the compilation were those of Schubert, Minding, Sohncke, Frenet, Graindorge, Brahy, Gregory, Roberts, and Carr. EDWARD L. DODD.

Annuaire du Bureau des Longitudes pour l'An 1909. Paris, Gauthier-Villars.

THE volume of the Annuaire for the current year is without the chemical and physical constants and certain astronomical tables which are inserted in the even-numbered years. As usual in each successive issue, some of the information is brought up to date; a significant change noted this year is the suppression of the interest tables for 5½ and 6 per cent., and the insertion of those for 1½ and 2 per cent.

The summaries of progress in special lines which we are accustomed to expect deal this year with variable stars by M. G. Bigourdan and with the movements and deformations of the crust of the earth by M. Ch. Lallemand, the former in 116, and the latter in 57 pages. Both of these articles are illustrated by numerous figures and really amount to brief treatises from which it is easy to obtain a fair idea of the theorems, methods and results in the two subjects. Two brief obituary notices of M. Janssen by MM. Radau and Deslandres, and a full alphabetical index close the volume. ERNEST W. BROWN.

NOTES.

THE twenty-fourth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on April 9–10.

THE Deutsche Mathematiker-Vereinigung now includes 725 members, of whom 60 are Americans. The Circolo Matematico di Palermo has a membership of 635, of whom 105 are Americans.

UNIVERSITY OF PARIS. The following mathematical courses, each two hours per week, are given in the second semester, beginning March 1: — By Professor E. PICARD: Certain developments in series occurring in analysis and mathematical physics. — By Professor E. GOURSAT: Differential equations. — By Professor P. PAINLEVÉ: General laws of motion of systems, analytic mechanics, hydrostatics and hydrodynamics. — By Professor P. APPELL: Elements of analysis and mechanics. — By Professor P. ANDOYER: Mathematical astronomy. — By Professor J. BOUSSINESQ: Thermodynamics and refrigeration. — By Professor G. KOENIGS: General theory of mechanisms.

Semiweekly conferences are held by Professors Raffy, Andoyer, and Servant.

The courses in the Ecole Normale are continuations of those of the first semester (see December BULLETIN, page 148).

UNIVERSITY OF MUNICH.— By Professor F. LINDEMANN: Analytic geometry of space, five hours; Definite integrals and Fourier series, four hours; Seminar, two hours.— By Professor A. VOSS: Integral calculus, five hours; Differential geometry of curved surfaces, II, three hours; Seminar, two hours.— By Professor A. PRINGSHEIM: Algebra, II, four hours; Applications of elliptic functions, three hours.— By Professor A. SOMMERFELD: Optics, four hours; Seminar, two hours.— By Professor H. BRUNN: Elements of higher mathematics for general students, three hours.— By Professor K. DOEHLEMANN: Descriptive geometry, II, with exercises, six hours; Geometric mechanics, with exercises, four hours.— By Dr. F. HARTOGS: Analytic functions of several variables, two hours. — By Dr. O. PERRON: Differential and integral calculus, with exercises, six hours.

UNIVERSITY OF STRASSBURG.— By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Algebraic numbers, two hours; Seminar, two hours.— By Professor F. SCHUR: General theory of curves and surfaces, four hours; Geometry of movement, two hours; Seminar, two hours.— By Professor J. WELLSTEIN: Introduction to the theory of finite discrete groups, four hours; Riemann surfaces, two hours; Seminar, two hours.— By Professor H. E. TIMERDING: Analytic geometry of space, four hours; Introduction to higher analytic geometry of the plane, four hours; Axonometry and perspective, two hours.— By Professor P. EPSTEIN: Harmonic functions and Bessel functions, four hours.

The following graduate courses in mathematics are announced for the year 1909–1910.

PRINCETON UNIVERSITY. (Each course is three hours per week.) — By Professor H. B. FINE: Advanced algebra (first term). — By Professor H. D. THOMPSON: Coordinate geometry. — By Professor J. H. JEANS: Mechanics and kinematics. —By Professor L. P. EISENHART: Differential geometry. — By Professor O. VEBLEN: Projective geometry, I; Projective geometry, II; Theory of functions of real variables. — By Professsor G. D. BIRKHOFF: Differential equations; Advanced calculus. — By Dr. E. SWIFT: Theory of functions of a complex variable.

Professor G. Bagnera, of the University of Messina, has accepted the professorship of analysis at the University of Palermo.

Professor G. Fubini, of the University of Genoa, has been appointed professor of mathematical analysis at the technical school at Turin.

Dr. R. v. Mises has been appointed docent in mechanics at the technical school at Brünn.

Dr. Krygowsky, of the technical school at Lemberg, has been promoted to an associate professorship of mathematics.

Professor J. Plemelj, of the University of Czernowitz, has been promoted to a full professorship of mathematics.

Professor H. Poincaré, of the University of Paris, has been elected honorary member of the academy of sciences at Vienna.

Dr. E. Buhl, of the University of Toulouse, has been promoted to a professorship of mathematics.

Professor M. Disteli, of the technical school at Dresden, has accepted the professorship of mathematics in the technical school of Carlsruhe made vacant by the resignation of Professor F. Schur, who was called to the University of Strassburg.

Professor Th. Reye, professor emeritus of the University of Strassburg, has been decorated with the order of the red eagle of the second class with the oaken wreath.

Professor Maurice Lévy, of the Collège de France, has been made professor emeritus.

Professor E. Landau, of the University of Berlin, has accepted a professorship of mathematics at the University of Göttingen, as successor to the late Professor H. Minkowski.

Dr. P. Carrus, of the University of Besançon, has been promoted to an associate professorship of mathematics.

Mr. D. D. Leib has been appointed instructor in mathematics at Yale University.

Professor G. Morera, of the University and technical school at Turin, died February 8, at the age of 53 years. He was a member of the academy of sciences of Turin, and of the royal academy dei Lincei.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bayes (T.). Versuch zur Lösung eines Problems der Wahrscheinlichkeitsrechnung. Herausgegeben von H. E. Timerding. (Ostwald's Klassiker der exakten Wissenschaften, 169.) Leipzig, Engelmann, 1908. 8vo. 59 pp. M. 1.20

Bôcher (M.). An introduction to the study of integral equations. Cambridge, University Press, 1909. 8vo. 8 + 72 pp. (Cambridge tracts in mathematics and mathematical physics, No. 10.) 2s. 6d.

Castells Vidal (P.). Resumen de las lecciones de análisis matemático, hasta las aplicaciones geométricas del cálculo diferencial. 3a edición. Barcelona, 1908. 737 pp. P. 35.00

Dölp (H.). Die Determinanten nebst Anwendung auf die Losung algebraischer und analytisch-geometrischer Aufgaben. Elementar behandelt. 7te Auflage. Darmstadt, Roether, 1908. 8vo. 4 + 95 pp. M. 2.00

Encyclopédie des sciences mathématiques pures et appliquées. Rédigée et publiée d'après l'édition allemande sous la direction de J. Molk. Tome I, Vol. 4. Calcul des probabilités; théorie des erreurs; applications diverses. Rédigé dans l'édition allemande sous la direction de F. Meyer. Fascicule 2. Paris et Leipzig, Teubner, 1908. 8vo. Pp. 161–320. Fr. 5.00

Hübner (H.). Ueber den Fermat'schen Satz. Erlangen, Junge, 1908. 8vo. 40 pp. M. 1.50

Mataix Araceli (C.). Resumen de las lecciones de cálculo integral y mecánica racional, explicadas en la Escuela Central de Ingenieros Industriales. Vols. I and II. Madrid, 1908. Vol. I, 351 pp.; Vol. II, 626 pp. P. 40.00

Meyer (F.). See Encyclopédie.

Molk (J.). See Encyclopédie.

Penkmayer (R.). Beweis des Satzes von Fermat: Die Gleichung $a^n + b^n = c^n$ ist in ganzen Zahlen unlösbar, wenn $n > 2$ ist. München, Lindauer, 1908. 8vo. 7 pp. M. 0.50

Timerding (H. E.). See Bayes (T.).

Weitzenböck (R.). Komplex-Symbolik. Eine Einführung in die analytische Geometrie mehrdimensionaler Räume. (Sammlung Schubert, LVII.) Leipzig, Goschen, 1908. 8vo. 6 + 191 pp. Cloth. M. 4.80

II. ELEMENTARY MATHEMATICS.

Amaldi (U.). See Enriques (F.).

Brumby (A. G.). Elementary practical geometry. London, Normal Press. 1909. 8vo. 214 pp. 1s. 6d.

Cours d'algèbre et de trigonométrie à l'usage des sous-officiers (élèves officiers). Rédigé conformément au programme de 1905. Paris, Charles-Lavauzelle, 1908. 16mo. 193 pp. Fr. 3.00

Dreyfus (L.). See Fort (L.).

Durell (C. V.). A course of plane geometry for advanced students. Part 1. London, Macmillan, 1909. 8vo. 232 pp. 5 s.

Enriques (F.) e Amaldi (L.). Elementi di geometria. 3a edizione ridotta. Bologna, Zanichelli, 1908. 16mo. 7 + 295 pp. L. 2.50

Fort (L.) et Dreyfus (U.). Éléments de géométrie, conformes aux programmes de 1905. Classe de 3e B. Avec de nombreux exercices. Paris, Paulin, 1909. 18mo. 201 pp. Fr. 2.50

Leder (E.). Die Praxis des Logarithmen-Rechnens unter Berücksichtigung der einschlägigen tabellarischen, graphischen und mechanischen Hilfsmittel. Ein Leitfaden für Kaufleute, Techniker, Gewerbetreibende, überhaupt für jeden praktischen Zahlenrechner. Berlin, Kühl, 1908. 8vo. 127 pp. M. 6.00

Lubarsky (L. H.). Elementary algebra: manual of examinations in mathematics; regents' questions and answers. New York, Engineering News Pub. Co., 1909. 8vo. 286 pp. Cloth. $1.50

Nociones elementales de geometría aplicadas al dibujo lineal; por los hermanos de las escuelas christianas. 13a edición. Paris, Procuraduria general, 1908. 16mo. 112 pp.

Paterson (W. E.). School algebra, Part 2. London, Frowde, 1909. 8vo. 292 pp. Cloth. 2s. 6d. With answers, 3 s.

——. School algebra. Complete. London, Frowde, 1909. 8vo. 604 pp. Cloth. 4 s. With answers, 5 s.

Riboni (G.). Elementi di geometria, ad uso delle scuole secondari e superiori. 5a edizione, con aggiunte e correzioni. Bologna, Zanichelli, 1908. 16mo. 531 pp. L. 3.00

III. APPLIED MATHEMATICS.

Abraham (M.). Theorie der Elektrizität. Vol. II. Elektromagnetische Theorie der Strahlung. 2te Auflage. Leipzig, Teubner, 1908. 8vo. 12 + 404 pp. Cloth. M. 10.00

Barucci (O.). Corso di disegno geometrico e ornamentale, per le scuole tecniche e complementari. Parte I. Torino, Paravia, 1909. 8vo. L. 1.60

Blanchin (P.). See Zenneck (J.).

Cramp (W.) and Smith (C. F.). Vectors and vector diagrams, applied to the alternating current circuit with examples of their use in the theory of transformers, and of single and polyphase motors, etc. New York, Longmans, 1909. 8vo. 16 + 252 pp. Cloth. $2.50

Einhorn (M.). Maturitäts-Aufgaben aus der darstellenden Geometrie, geeignet als Uebungs- und Repetitions-Aufgaben für die oberen Klassen der Realschule und für die Vorbereitung zur Aufnahme-Prüfung am Eidgenössischen Polytechnikum. 1tes Heft. Zürich, Speidel, 1908. 8vo. 30 pp. M. 0.70

Föppl (A.). Vorlesungen über technische Mechanik. (In 6 volumes.) Vol. 4: Dynamik. 3te, stark veränderte Auflage. Leipzig, Teubner, 1909. 8vo. 8 + 422 pp. Cloth. M. 10.00

Gauss. See Lagrange.

Gibson (A. H.). Hydraulics and its application. New York, Van Nostrand, 1908. 8vo. 16 + 757 pp. Cloth. $5.00

GUÉRARD (G.). See ZENNECK (J.).

HANCOCK (E. L.). Applied mechanics for engineers: a textbook for engineering students. New York, Macmillan, 1909. 12mo. 11 + 385 pp. Cloth. $2.50

JACOBI. See LAGRANGE.

JOURDAIN (P. E. B.). See LAGRANGE.

LAGRANGE, RODRIGUES, JACOBI und GAUSS. Abhandlungen über die Prinzipien der Mechanik, herausgegeben von P. E. B. Jourdain. (Ostwald's Klassiker der exakten Wissenschaften, 167.) Leipzig, Engelmann, 1908. 8vo. 68 pp. M. 1.40

PICOT (E.). See ZENNECK (J.).

RODRIGUES. See LAGRANGE.

SMITH (C. F.). See CRAMP (W.).

STEINMETZ (C. P.). Theory and calculation of alternating current phenoména. 4th edition. London, Spon, 1909. 8vo. Cloth. 21 s.

THUET (J.). Cours de mécanique à l'usage des élèves des classes de mathématiques A et B et des candidats à l'École navale et à l'Ecole de Saint-Cyr (programme de 1905). Paris, Belin, 1908. 8vo. 360 pp. Fr. 4.50

ZENNECK (J.). Les oscillations électromagnétiques et la télégraphie sans fil. Ouvrage traduit de l'allemand par P. Blanchin, G. Guérard, et E. Picot. Vol. 2: les oscillations ouvertes et les systèmes couplés; les ondes électromagnétiques et la télégraphie sans fil. Paris, Gauthier-Villars, 1908. 8vo. 496 pp. Fr. 17.00

THE FEBRUARY MEETING OF THE SAN FRANCISCO SECTION.

THE fifteenth regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY was held at Stanford University, Saturday, February 27, 1909. The following fourteen members of the society were present:

Professor R. E. Allardice, Professor H. F. Blichfeldt, Professor R. L. Green, Professor M. W. Haskell, Professor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Professor C. A. Noble, Mr. H. W. Stager, Professor Irving Stringham, Mr. J. D. Suter, Professor S. D. Townley, Professor A. W. Whitney.

Two sessions were held, a morning session opening at 10:30 A. M. and the afternoon session at 3:00 P. M., after a luncheon at which all those attending the meeting were present.

The following papers were read at this meeting:

(1) Professor A. W. WHITNEY. "An algebra of probable inference" (preliminary communication).

(2) Professor R. E. ALLARDICE. "Note on the reduction of a circulant."

(3) Professor R. E. ALLARDICE. "A theorem in the partition of numbers."

(4) Professor W. A. MANNING. "On the order of primitive groups. Second memoir."

(5) Mr. G. F. MCEWEN. "Forced vibrations of a pendulum in a viscous fluid."

(6) Mr. H. W. STAGER. "Some investigations in the theory of numbers."

Mr. McEwen was introduced by Professor H. C. Moreno. Abstracts of the papers follow below in the order of their presentation.

1. The paper by Professor Whitney points out the unsatisfactory character of the fundamental conceptions of the theory of probability. The resolution of the difficulty is believed to lie in the recognition of the theory of probability as a system of relations which have their origin in the fundamental laws of thought. The nature of the particular judgments in this field

have the subsidiary importance that similar particular judgments hold in the logic of certain inference, where the dissociability of the formal problem from the objective problem is recognized; in the theory of probability this distinction has been obscure.

This body of relations, which properly forms the theory of probability, is obtained by generalizing the algebra of the logic of classes so as to include the implicatory relations of classes; this is done by the introduction of a symbol, AB/A, the implicatory or subsumptive relation of the class A to the class AB, and certain defining operations.

Ordinary formal logic, or the algebra of certain inference, is included as a special case, and it is believed that this generalization serves to give expression as well to the algebra of propositions.

The algorithm gives, as an example, the following expression for a simple case of inverse probability, BA/B, that is the probability, if the event B has happened, of the hypothesis

$$A\,.\,BA/B\cdot\frac{A\cdot AB/B}{A\cdot AB/A+\bar{A}\cdot\bar{A}B/A}.$$

2. In his note on the reduction of a circulant, Professor Allardice showed that the circulant $C(x, y, z, 0, 0, \cdots)$, of prime order $2p+1$, may be expanded in the form

$$\sum x^{2p+1}+\sum_{r=1}^{r=p}\left[(-)^r\frac{2p+1}{2p-2r+1}\underset{2p-r\;r}{C}\,y^{2p-2r+1}(xz)^r\right].$$

3. Professor Allardice's theorem in the partition of numbers is as follows: If $2p+1$ is a prime, the number of r-partitions of all numbers congruent to k (mod $2p+1$) by means of numbers different from one another and not greater than $2p+1$ is

$$\frac{1}{2p+1}\cdot\underset{2p+1\;r}{C};$$

and the number of double partitions of the same type is

$$\frac{p}{(2p+1)^2}\cdot\left[\frac{1}{2p}\underset{2p+1\;2r}{C}\times\underset{2r\;r}{C}+(-)^r\frac{2p+1}{2p-2r+1}\underset{2p-r\;r}{C}\right].$$

4. Professor Manning's paper is a continuation of an article in the April number of the *Transactions* under the same title. In that paper a theorem announced without proof by Camille Jordan is proved for certain cases and somewhat extended. The proof is now completed by showing that a non-alternating primitive group of degree $5p + k$, where p is a prime number greater than 5, and k is an integer greater than 6, cannot contain a permutation of order p on five cycles. An easy deduction from these results is the theorem:

If the order of a primitive group of degree n exceeds

$$n!/p_1 \cdots p_2^2 \cdots p_3^3 \cdots p_4^4 \cdots p_5^5 \cdots 3^5\ 2^5,$$

the group is either alternating, symmetric, or one of a small number of groups of low class.

The numbers $p_1, \cdots, p_2, \cdots$ are the primes less than $n - 2$ written in descending order. The conditions

$$p_2 < \frac{n}{2} - 1,\ p_3 < \frac{n}{3} - 1,\ p_4 < \frac{n}{4} - 1,\ p_5 < \frac{n-1}{5} - 1,$$

must be satisfied by primes which enter the denominator to a higher power than the first.

5. Mr. McEwen gave an account of a mathematical investigation, with accompanying physical experiments, of the motion in a viscous fluid of a sphere attached to a compound gravity pendulum. Upon the support of the pendulum is impressed the horizontal motion

$$x_0 = ce^{-at} \sin at.$$

When the effect on the pendulum of both the air and the fluid is considered, the differential equation of motion is

$$P\frac{d^2x}{dt^2} + P_1\frac{dx}{dt} + P_2 x + N = P_0''ce^{-at} \sin at + P_0'ce^{-at} \cos at.$$

The coefficients $P, P_1, \cdots$ involve μ, the coefficient of viscosity of the fluid, and known constants only. The particular integral that expresses the forced vibration is

$$x = ce^{-at}[A \sin (at - \theta_1) + B \cos (at - \theta_1)] - \frac{N_2}{P},$$

where

$$\tan\theta_1 = \frac{P_1 a - 2a\alpha P}{P_2 + P(\alpha^2 - a^2) - \alpha P_1} = \tan\left(at_1 - \tan^{-1}\frac{P'_0}{P''_0}\right),$$

and in which t_1 is the phase difference in seconds between the motion of the sphere and that of the support.

6. The particular class of numbers which are discussed in Mr. Stager's paper suggested themselves in the construction of a table for use in applying Sylow's theorem. This table — now constructed for the first 10,000 numbers and eventually to be extended to 15,000 or 20,000 — exhibits the value of k for all prime factors of each number which give factors of the form $p(kp + 1)$, where p is any prime except 2 and k is any positive integer. Those numbers, called P's, which contain no factor of the form $p(kp + 1)$ form a very interesting class. The present paper deals with their fundamental properties; shows that the numbers consist of four general types; obtains several formulas for their enumeration; and suggests a connection between the number of ordinary primes and the number of P's within a given limit.

W. A. MANNING,
Secretary of the Section.

THE CONSTRUCTION OF A SPACE FIELD OF EXTREMALS.

BY DR. E. GORDON BILL.

(Read before the American Mathematical Society, December 30, 1908.)

IT is a well-known theorem of the calculus of variations,[*] that if all the members of a one parameter family of plane curves pass through a fixed point O, then any arc of these extremals which does not contain O nor its conjugate point, may be imbedded in a field.

Moreover, in 1879 Weierstrass stated that a field including O could be constructed and Professor Bliss[†] has proved this to be true.

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[*] Osgood, *Annals of Math.*, ser 2, vol. 2, p. 112.

[†] Bliss, BULLETIN, vol. 13, p. 321.

It is equally true, that if we have a two-parameter family of space curves, through a fixed point O, we can construct a space field of extremals including O in its interior.

In order to prove the above statement, we assume O to be an ordinary point of the problem, and choose it for the origin of coordinates.

The extremals * are given by the equations

$$(1)\quad x=\phi(s,\alpha,\beta,\gamma),\quad y=\psi(s,\alpha,\beta,\gamma),\quad z=\chi(s,\alpha,\beta,\gamma),$$

where ϕ, ψ, χ are of class C'' for s in $(0\bar{S})$ and $\alpha^2+\beta^2+\gamma^2=1$, s being the length of arc measured along an extremal from O, and α, β, γ the direction cosines of the tangents to the extremals at O. Under these assumptions the following relations † evidently exist:

$$(2)\quad \begin{aligned}
\phi(0)=\psi(0)&=\chi(0)=0,\\
\phi_s(0)=\alpha,\ \psi_s(0)=\beta,\ \chi_s(0)&=\gamma,\\
\phi_{s\alpha}(0)=1,\ \psi_{s\beta}(0)=1,\ \chi_{s\gamma}(0)&=1,\\
\phi_\alpha(0)=\phi_\beta(0)=\phi_\gamma(0)&=0,\\
\psi_\alpha(0)=\psi_\beta(0)=\psi_\gamma(0)&=0,\\
\chi_\alpha(0)=\chi_\beta(0)=\chi_\gamma(0)&=0,\\
\phi_{s\beta}(0)=\phi_{s\gamma}(0)=\psi_{s\alpha}(0)&=0,\\
\phi_{s\gamma}(0)=\chi_{s\alpha}(0)=\chi_{s\beta}(0)&=0.
\end{aligned}$$

In order to establish the existence of the above field, it will be sufficient to prove the following:

THEOREM A. *There exists a sphere S having its center at O, such that for every point in it s, α, β, γ are defined as one-valued functions of x, y, z of class C', i. e., the equations*

$$(3)\quad \begin{aligned}
F_1&\equiv x-\phi(s,\alpha,\beta,\gamma)=0, & F_2&\equiv y-\psi(s,\alpha,\beta,\gamma)=0,\\
F_3&\equiv z-\chi(s,\alpha,\beta,\gamma)=0, & F_4&\equiv \alpha^2+\beta^2+\gamma^2-1=0,
\end{aligned}$$

have one and only one solution $(s, \alpha, \beta, \gamma)$ corresponding to any set of values $(x; y, z)$ in S.

Lemma I. There exists a quantity $S'\leqq\bar{S}$, such that for s in $(0S')$, $[0<s\leqq S']$, the radius vector r of any point on the extremals is an increasing function of s.

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* Bliss and Mason, "Properties of curves in space which minimize a definite integral," *Transactions Amer. Math. Society*, vol. 9, no. 4.

† $\phi(0)=\phi(0,\alpha,\beta,\gamma)$, etc.

We have r defined as a function of s by $r^2 = \phi^2 + \psi^2 + \chi^2$; hence

$$\frac{dr}{ds} = \frac{\phi\phi_s + \psi\psi_s + \chi\chi_s}{\sqrt{\phi^2 + \psi^2 + \chi^2}};$$

but by Taylor's formulas

$$\phi = s\phi_s(\theta_1), \quad \psi = s\psi_s(\theta_2), \quad \chi = s\chi_s(\theta_3),$$

and thus by relations (2)

$$\lim_{s=0} \frac{dr}{ds} = \frac{\alpha^2 + \beta^2 + \gamma^2}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}} = 1.$$

Hence, from the continuity of r, the above S' exists and the surface of a sphere of radius

$$r' = min \sqrt{\phi(S')^2 + \psi(S')^2 + \chi(S')^2}$$

will be cut once and only once by each extremal, for a value of s in $(0 \cdot S')$.

Lemma II. There exists a quantity $S'' \leqq \bar{S}$ such that for s in $(0 \cdot S'')$ the Jacobian Δ of equations (3) does not vanish.

We have

$$\frac{\Delta}{2} = \begin{vmatrix} \phi_s & \phi_\alpha & \phi_\beta & \phi_\gamma \\ \psi_s & \psi_\alpha & \psi_\beta & \psi_\gamma \\ \chi_s & \chi_\alpha & \chi_\beta & \chi_\gamma \\ 0 & \alpha & \beta & \gamma \end{vmatrix}$$

and hence from (2) $\Delta(0, \alpha, \beta, \gamma) = 0$; but expanding $\frac{1}{2}\Delta$ in terms of the elements of the last row and applying Taylor's formulas, we have from (2)

$$\lim_{s=0} \frac{\Delta}{2s^2} = \alpha \begin{vmatrix} \alpha & 0 & 0 \\ \beta & 1 & 0 \\ \gamma & 0 & 1 \end{vmatrix} - \beta \begin{vmatrix} \alpha & 1 & 0 \\ \beta & 0 & 0 \\ \gamma & 0 & 1 \end{vmatrix} + \gamma \begin{vmatrix} \alpha & 1 & 0 \\ \beta & 0 & 1 \\ \gamma & 0 & 0 \end{vmatrix}$$

$$= \alpha^2 + \beta^2 + \gamma^2 = 1.$$

Thus in the neighborhood of O

$$\Delta = 2s^2[1 + E(s)] \quad \text{where} \quad \lim_{s=0} E(s) = 0,$$

hence S'' exists, and about O there is a sphere with radius

$$r'' = min \sqrt{\phi(S'')^2 + \psi(S'')^2 + \chi(S'')^2},$$

such that for all points in it, except O, the Jacobian of equations (3) is not zero.

Let us choose $r > 0$ to be less than r' and r''. We wish to show that the sphere S of radius r about O satisfies the conditions of Theorem A.

We are assured by Lemma I that for s in a restricted interval, the surface of S is cut once and only once by each extremal through O, and by Lemma II we know that for each point of the surface of S, the Jacobian of (3) does not vanish.

Next, let $P_0(x_0, y_0, z_0)$ be a point where one of the extremals (1) pierces the surface of S. The values of s, α, β, γ corresponding to x_0, y_0, z_0 will satisfy (3), and as the functions F_i $(i = 1, \cdots, 4)$ admit first derivatives in the vicinity of P_0, and as $\Delta(P_0) \neq 0$, we can state by the well known theorem on implicit functions that:

In a certain zone Z_0 of S about P_0 there exist one-valued functions $\bar{s}(x, y, z)$, $\bar{\alpha}(x, y, z)$, $\bar{\beta}(x, y, z)$, and $\bar{\gamma}(x, y, z)$ of class C' which reduce to s_0, α_0, β_0, γ_0 at (x_0, y_0, z_0) and which identically satisfy (3) in x, y, z in Z_0.

The remainder of our problem is to extend the domain of definition of these functions into larger and larger zones of S until finally they are defined for the entire sphere S.

First we will extend the domain of definition around the base B_0 of Z_0.

Take P_1 on B_0 (Fig. I); then as above in a sphere S'_1 about P_1, functions s, α, β, γ are defined satisfying conditions required and being identical in Z_0 with $\bar{s}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$.

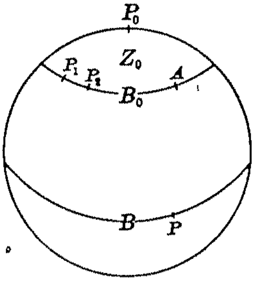

FIG. I.

Next take P_2, the intersection of S'_1 and B_0; then in S'_2 about P_2, functions s, α, β, γ are defined, being identical with $\bar{s}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ in Z_0 and S'_1.

Thus, if by means of extension spheres S'_i we can extend the domain of definition of functions s, α, β, γ completely around B_0, we are assured of coming back to P_1, with the values s_1, α_1, β_1, γ_1 originally defined there; for each extension sphere S'_i

overlaps Z_0 and S'_{i-1} and in these common parts, the functions s, α, β, γ are identical with those originally defined.

Now, certainly we can extend completely around B_0 by the above process, provided the lower limit of the radii of the extension spheres S'_i is not zero. If, however,

$$\text{lower limit } r'_i = 0,$$

the region of extension will be limited at some point A (Fig. I) on B_0; but the functions s, α, β, γ are limited near A and hence must have at least one set of limiting values $(\bar{s}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$. Moreover, s, α, β, γ are continuous in S and hence $\bar{s}$, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are solutions of equations (3), and thus as above we can extend past $(\bar{s}, \bar{\alpha}, \bar{\beta}, \bar{\gamma})$; hence the region of extension is not limited at A and therefore

$$\text{lower limit } r_i \neq 0.$$

Let us now replace all the extension spheres S'_i by a sphere S'_m of fixed radius r'_m,

$$0 < r'_m \leqq \text{lower limit } r'_i.$$

By means of S'_m we can extend the region of definition of s, α, β, γ from Z_0 into Z_1, a larger zone of S.

Similarly we can extend from Z_1 to Z_2 to Z_3, etc., until finally $s(x, y, z)$, $\alpha(x, y, z)$, $\beta(x, y, z)$ and $\gamma(x, y, z)$ are defined for all points of S, provided, however, the radii r^i_m of the uniform extension spheres S^i_m do not converge to zero.

If
$$\text{lower limit } r^i_m = 0$$

the zones Z_0, Z_1, Z_2, etc., will be limited in extent by the base B (Fig. I) of a zone Z and beyond this we cannot extend.

But since B is the limiting position of a series of bases B_0, B_1, B_2, etc., and since s, α, β, γ remain limited, they must as above approach definite limiting values $(s, \alpha, \beta, \gamma)$ as we approach a point P of B.

However s, α, β, γ are continuous functions in the region considered, hence $(s, \alpha, \beta, \gamma)$ are solutions of equations (3) and thus we can extend past P, *i. e.*, past B.

So, finally, for every point of S, s, α, β, γ are defined as one-valued functions of x, y, z of class C', *i. e.*, a field has been constructed about O.

YALE UNIVERSITY,
January 22, 1909.

THE SECOND VARIATION OF A DEFINITE INTEGRAL.

BY DR. A. L. UNDERHILL.

(Read before the Chicago Section of the American Mathematical Society, January 1, 1908, and January 2, 1909.)

Introduction.

IN the discussion of the minimizing of

$$J = \int_{t_0}^{t_1} F(x, y, x', y')dt \tag{1}$$

it may happen that the solution of the Jacobi equation presents considerable difficulty. In the following pages a method is given whereby without the solution of the Jacobi equation certain information may be obtained as to the extent of the interval in which a weak minimum exists.

It will be shown that by the evaluation of a quantity $\overline{K}_0$, which is computed along the extremal of the problem, one can draw the following conclusions:

1) If $\overline{K}_0 \leqq 0$ along a given extremal arc, there exists no conjugate point on the arc, *i. e.*, the extremal furnishes a weak minimum.

2) If $0 < 1/\overline{K}_0 \leqq a^2$ along an extremal, then the extremal cannot be a minimizing curve in any arc along which the value of the integral J is greater than πa.

3) If $0 < b^2 \leqq 1/\overline{K}_0$, then the extremal furnishes a weak minimum on any arc along which the value of J is less than or equal to πb.

§ 1. *Proofs of the Preceding Theorems.*

During a study of invariants connected with the minimizing of (1), it was noted by the writer that the second variation of J could be given the following invariantive normal form:*

$$\delta^2 J = \int_{\alpha_0}^{\alpha_1} \left[\left(\frac{dV}{d\alpha} \right)^2 - \overline{K}_0 V^2 \right] d\alpha, \tag{2}$$

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[*] *Transactions Amer. Math. Society*, vol. 9 (1908), p. 336.

in which

$$a = \int_{t_0}^{t} F(x, y, x', y')dt,$$

and V and $\overline{K}_0$ are two absolute invariants under a point transformation

$$x = X(u, v), \quad y = Y(u, v),$$

as well as a parameter transformation $t = \chi(\tau)$; the defining equations for V and $\overline{K}_0$ are

$$V = \omega F^{\frac{1}{2}} = wF^{\frac{1}{4}} F_1^{\frac{1}{4}} = (y'\delta x + x'\delta y)F^{\frac{1}{4}} F_1^{\frac{1}{4}} \,*$$

$$(3) \qquad \overline{K}_0 = \frac{1}{F^{\frac{1}{2}}}\left[\frac{1}{4}\frac{F_1'^2}{F_1^2} - \frac{1}{2}\frac{F_1''}{F_1} - \frac{F_2}{F_1} - \frac{1}{2}\frac{F''}{F} + \frac{3}{4}\frac{F'^2}{F^2}\right]. \,\dagger$$

From the above form (2) of the second variation follows at once:

THEOREM A: *In case* $\overline{K}_0 \leqq 0$ *along an extremal arc, it follows that* $\delta^2 J > 0$, *and the extremal arc furnishes a weak minimum.*‡

In order to obtain the other results, we start from (2) and form its Jacobi equation §

$$(4) \qquad \psi(V) = -\overline{K}_0 V - \frac{d^2 V}{da^2} = 0$$

and denote by $V = \Phi(\alpha)$ the solution of the equation which vanishes for $\alpha = \alpha_0$. The next zero of $\Phi(\alpha)$ may often be approximately located by comparing it with the solution of the equation

$$(5) \qquad \frac{d^2 V}{d\alpha^2} + \frac{1}{a^2} V = 0$$

which vanishes at $\alpha = \alpha_0$,

$$V = c \sin\left(\frac{\alpha - \alpha_0}{a}\right).$$

The next zero of this solution is at $\alpha = \alpha_0 + \pi a$.

Use may now be made of the following theorems of Sturm,‖

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* *Transactions*, loc. cit., p. 336, p. 329, p. 327.

† Ibid., p. 334.

‡ Ibid., p. 336.

§ Cf. Bolza, Lectures on the calculus of variations, p. 133.

‖ Darboux, Théorie des surfaces, § 629.

regarding the solutions of the two differential equations

$$\frac{d^2V}{dx^2} = HV, \quad \frac{d^2V}{dx^2} = H'V. \tag{6}$$

1) If $V = \phi(x)$ is a solution of the first equation having consecutive zeros at x_0 and x_1, and if $H' \geqq H$ for all values of x in the interval (x_0, x_1), then the solution of the second which vanishes at x_0 does not vanish again within or at the end point x_1 of the interval (x_0, x_1).

2) If $H' \leqq H$ for all values of x in (x_0, x_1), then the solution of the second which vanishes for $x = x_0$ has at least one zero in the interval (x_0, x_1).

Using (5) and (4) as the two equations of (6), we can formulate the following:

THEOREM B: *If* $1/\overline{K}_0$ *is positive and greater than* b^2 *along an extremal, then any arc of the extremal along which the value of* J *is* $\leqq \pi b$ *furnishes a weak minimum to* J.

THEOREM C: *If* $1/\overline{K}_0$ *is positive and less than* a^2 *along the extremal, then no arc of the extremal along which the value of* J *is greater than* πa *can be a minimizing curve.*

Comparing these theorems, we notice that A is applicable in case $\overline{K}_0$ is $\leqq 0$, while B and C apply when $\overline{K}_0$ is $\geqq 0$, the former when $1/\overline{K}_0$ is greater, the latter when $1/\overline{K}_0$ is smaller, than some definite constant.

§ 2. *Applications.*

Since $\overline{K}_0$ is an invariant under both point and parameter transformation the computational work is simplified by properly choosing the coordinate system as well as the parameter of the extremal.

1) *The geodesic problem.**

We choose as the u, v coordinates of the surface the geodesic parallel coordinates and for parameter the arc of the curve on the surface. $\overline{K}_0$ turns out to be the gaussian curvature, and we have, when use is made of theorem A, the well known Jacobi-Bonnet theorem:

On a surface of negative curvature, a given point has no conjugate point on the geodesics which pass through it.

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* Result already given in *Transactions*, loc. cit., p. 337.

2) *The problem of the shortest distance between two points.*

By choosing the arc as the parameter, the extremal may be written in the form

$$x = \alpha + \beta s, \quad y = \gamma + \delta s,$$

α, β, γ, and δ being constants.

The computation of the several quantities which appear in $\overline{K}_0$, each one being taken along the extremal, leads to the following:

$$F = 1, \quad F' = 0, \quad F'' = 0,$$
$$F_1 = 1, \quad F_1' = 0, \quad F_1'' = 0,$$
$$F_2 = x'x''' + y'y''' = 0.$$

As a result, $\overline{K}_0 = 0$ and $1/\overline{K}_0 = \infty > b^2$ for any finite b, and by means of theorem B we have:

For the problem of finding the shortest distance between two points, the extremal

$$x = \alpha + \beta s, \quad y = \gamma + \delta s,$$

furnishes a weak minimum in any interval.

3) *The problem of the minimum surface of revolution.*

The extremal is in this case a catenary and we choose the coordinate system so that the curve is symmetrical with respect to the y-axis, with its lowest point at $(0, a)$. If the arc s is selected as the parameter, then the equation of the extremal is

$$x = a \operatorname{arcsinh}\left(\frac{s}{a}\right). \quad y = \frac{a}{2}\left[e^{\operatorname{arcsinh}\left(\frac{s}{a}\right)} + e^{-\operatorname{arcsinh}\left(\frac{s}{a}\right)}\right],$$

since $s = a \sinh(s/a)$.

The computation leads to the following:

$$F = y, \quad F' = y', \quad F'' = y'',$$
$$F_1 = y, \quad F_1' = y', \quad F_1'' = y'',$$
$$F_2 = -y'' - y(x'y'' - x''y')^2 = -y'' - \frac{y}{\rho^2},$$

where ρ is the radius of curvature for the catenary. Since $1/\rho = a/y^2$, one obtains upon substitution of these values in $\overline{K}_0$

$$\bar{K}_0 = \frac{1}{y^4} \leqq \frac{1}{a^4}, \quad \frac{1}{\bar{K}_0} \geqq a^4,$$

since $y \geqq a$.

The use of theorem B leads to the following statement:

For the problem of the minimum surface of revolution, an arc of the catenary along which the value of J is at most equal to πa^3 furnishes a weak minimum.

In case (x_0, y_0) is the first point of the arc of the extremal, the abscissa $\bar{x}$ of the end point of the arc which will satisfy the above condition is given by the equation

$$\sinh\left(\frac{2x}{a}\right) + 2\bar{x} - \left[\sinh\left(\frac{2x_0}{a}\right) + 2x_0\right] = 4\pi a.$$

4) *The brachistochrone problem.*

The extremal is a cycloid and the origin is chosen at a cusp, the y axis is directed downward, and the arc s is selected as the parameter.

From the usual equation of the cycloid by means of the relation

$$s = 4a(1 - \cos \tfrac{1}{2}\theta)$$

we obtain the equation of the extremal under the present choice of axis and parameter as

$$x = a\left[\sin^{-1}\left(2\sqrt{\frac{s}{2a} - \frac{s^2}{16a}\cdot\frac{4a-s}{4a}}\right) - 2\left(2\sqrt{\frac{s}{2a} - \frac{s^2}{16a}\cdot\frac{4a-s}{4a}}\right)\right],$$

$$y = s - \frac{s^2}{8a}.$$

The computation gives the following values:

$$F = \frac{1}{\sqrt{y}}, \quad F' = -\frac{1}{2}\frac{y'}{\sqrt{y^3}}, \quad F'' = \frac{3}{4}\frac{y'^2}{\sqrt{y^5}} - \frac{1}{2}\frac{y''}{\sqrt{y^3}},$$

$$F_1 = \frac{1}{\sqrt{y}}, \quad F_1' = -\frac{1}{2}\frac{y'}{\sqrt{y^3}}, \quad F_1'' = \frac{3}{4}\frac{y'^2}{\sqrt{y^5}} - \frac{1}{2}\frac{y''}{\sqrt{y^3}},$$

$$F_2 = \frac{3}{4}\frac{1}{\sqrt{y^5}} + \frac{1}{2}\frac{y''}{\sqrt{y^3}} - \frac{3}{4}\frac{y'^2}{\sqrt{y^5}} - \frac{1}{\rho^3\sqrt{y}}.$$

For the cycloid

$$\frac{1}{\rho^2} = \frac{1}{16a^2 \sin^2 \frac{1}{2}\theta} = \frac{1}{8a\left(s - \dfrac{s^2}{8a}\right)} = \frac{1}{8ay},$$

and after substitution of these values in $\overline{K}_0$ we find

$$\overline{K}_0 = \frac{1}{8ay}\{2a(y'^2 - 3) + y\},$$

or making use of the extremal equation,

$$\overline{K}_0 = -\frac{1}{2y}, \quad i.\ e., \quad \overline{K}_0 < 0.$$

By means of theorem A, we have the result:

For the brachistochrone problem there is no conjugate point to any point P lying on the same cycloid arch with P.

THE UNIVERSITY OF WISCONSIN.

A SIMPLER PROOF OF LIE'S THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS.

BY PROFESSOR L. D. AMES.

(Read before the Chicago Section of the American Mathematical Society, April 9, 1909.)

THE following theorem is essentially equivalent to Lie's principal theorem concerning the integration of the differential equation $\Omega(x, y, y') = 0$ when it is invariant under a known group. As stated here, this theorem makes no use of the idea of a group.

THEOREM. *Given any differential equation of the form*

$$(1) \qquad \Omega(x, y, y') = 0$$

which can be solved in the form

$$(2) \qquad X(x, y)y' - Y(x, y) = 0;$$

if $\xi(x, y)$ and $\eta(x, y)$ are such functions that

$$X\eta - Y\xi \not\equiv 0,$$

then a necessary and sufficient condition that

$$(3)\qquad \frac{X}{X\eta - Y\xi}\,Y' - \frac{Y}{X\eta - Y\xi} = 0$$

be exact is that

$$(4)\quad \xi\frac{\partial\Omega}{\partial x} + \eta\frac{\partial\Omega}{\partial y} + \left(\frac{\partial\eta}{\partial x} + y'\frac{\partial\eta}{\partial y} - y'\frac{\partial\xi}{\partial x} - y'^2\frac{\partial\xi}{\partial y}\right)\frac{\partial\Omega}{\partial y'} = 0.^*$$

Proof. Let us first assume that the given equation is in the solved form

$$(5)\qquad \Omega_1(x, y, y') \equiv X(x, y)y' - Y(x, y).$$

A necessary and sufficient condition that (3) be exact is that

$$(6)\qquad \frac{\partial}{\partial x}\left(\frac{X}{X\eta - Y\xi}\right) + \frac{\partial}{\partial y}\left(\frac{Y}{X\eta - Y\xi}\right) = 0.$$

If we expand (6), drop the denominator, collect, change signs, divide by X, and put $Y/X = y'$ from (2), we obtain the required condition (4). All these steps are possible since $X\eta - Y\xi \not\equiv 0$ by hypothesis, and $X \not\equiv 0$.

Next let the given equation be any equation of the form

$$\Omega(x, y, y') = 0$$

which can be solved in the form

$$\Omega_1(x, y, y') \equiv Xy' - Y = 0$$

throughout a region R in the three variables x, y, y', thought of as coordinates of a point in three dimensions. Differentiating each, we obtain

$$\frac{\partial\Omega}{\partial x}\,dx + \frac{\partial\Omega}{\partial y}\,dy + \frac{\partial\Omega}{\partial y'}\,dy' = 0,$$

$$\frac{\partial\Omega_1}{\partial x}\,dx + \frac{\partial\Omega_1}{\partial y}\,dy + \frac{\partial\Omega_1}{\partial y'}\,dy' = 0.$$

It follows that

$$\frac{\dfrac{\partial\Omega}{\partial x}}{\dfrac{\partial\Omega_1}{\partial x}} = \frac{\dfrac{\partial\Omega}{\partial y}}{\dfrac{\partial\Omega_1}{\partial y}} = \frac{\dfrac{\partial\Omega}{\partial z}}{\dfrac{\partial\Omega_1}{\partial z}}.$$

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[*] In the language of the Lie theory this is the condition that $\Omega(x, y, y') = 0$ be invariant under the group $Uf = \xi(\partial f/\partial x) + \eta(\partial f/\partial y)$. See Lie-Scheffers, Vorlesungen über Differentialgleichungen mit bekannten infinitesimalen Transformationen, p. 97.

Hence the results obtained by using Ω and Ω_1 respectively in (4) differ only by a factor, and the condition (4) applies equally well to the equation in the general form. This proves the theorem as stated.

The Lie theory gives no general method of finding the group of a given differential equation, and the problem of finding such a group is precisely the problem of finding ξ and η, solutions of (4).

If ξ and η are chosen in advance, the general form of the differential equation which can be integrated by their aid can be determined in precisely the same manner as in the Lie theory. Thus the Lie theory offers no formal aid in the solution of the above mentioned type of equation not furnished by this theorem.

However, there are some interesting considerations of a geometrical character due to Monge, Lie, and others which are suggestive occasionally; it is possible to restate some of these also, and the author proposes to do this in a later paper.

The method used in this paper can be applied to other types of differential equations, and Lie's essential results for those types obtained. These types will form the subject matter of another paper.

UNIVERSITY OF MISSOURI,
March 3, 1909.

HEATH'S EUCLID.

The Thirteen Books of Euclid's Elements Translated from the Text of Heiberg with Introduction and Commentary. By T. L. HEATH, C.B., Sc.D. Cambridge, University Press, 1908. 8vo. 3 volumes. $13.50.

IT may properly be said of Dr. Heath's latest contribution to mathematical literature — and may it not be his last! — that it is characteristically British, and perhaps nothing better could be said of a work on the great mathematical classic of the world. We are tempted to think that England does not produce the great works of Germany, say in the history of mathematics as also in the theory of this science, and we point to Cantor and to Hankel, to Braunmühl and possibly to Günther, or even to Simon, as justifying this thought. And yet, when the English scholar does bring out a book, it is quite as when a British general takes possession of a conquered province — there is nothing more to say. What so scholarly work has appeared on early Greek geometry as Allman's, or where is there such a readable survey of Greek mathematics as a whole as is to be found in Gow? What elementary history of mathematics is so charmingly written as Ball, or what one has found its way into so many languages or has passed through so many editions?

And the same questions may be asked with respect to the ancient classics of mathematics — where may one go save to England? Heiberg's scholarly contributions? — yes, and the world is greatly the debtor of this indefatigible worker whose ability to read the medieval manuscripts of mathematical classics is probably not equaled among living scholars. But Heiberg is working on an entirely different problem, one of text criticism, of finding the probable form of the original, and of comparing the codices. When it comes to a thorough study of the mathematical side of the classics of Greece, the world must go to England. Paul Tannery might have done this kind of work if he had only had the time, although his great scholarship left a monument built merely of fragments. But no continental writer has stayed with the problem with the tenacity of purpose that characterizes a Briton when he sets to work at his task, and hence it is to England that we must go for Diophantos, for Apollonius, for Archimedes, and for Euclid — to England where Greek is not yet dishonored, and where the classics in mathematics are still held in highest esteem.

But when we consider what scholars have made possible this appreciation of the works of the ancient masters, when we seek out the names of those who have followed in the footsteps of Billingsley and Wallis and Simson, we find ourselves possessed of doubt as to whether it is England after all that produces such works as these, for we find that it is one single scholar who has done it all. Cantor has written a monumental work, and Zeuthen has contributed worthily to the science, and the lamented Braunmühl has given us a helpful and scholarly treatise, and Loria has written with his usual brilliancy,— but after all, it is the kind of work that Dr. Heath has done that makes the work of these other men possible. And so, whether it be the effect of English education, or British tenacity, or the individual writer, all who are interested in the history of mathematics should recognize their indebtedness and the indebtedness of education in general, to the man who has produced these works.

Dr. Heath's Diophantos was published in 1885, a single volume of 248 pages, and the first noteworthy modern treatment of the work of the father of Greek algebra. This was followed in 1896 by a volume on Apollonius of Perga, and this in 1897 by the works of Archimedes. And now, after more than ten years, and as a culmination of these contributions, comes the Euclid, a work that no one but an Englishman could write in the true *con amore* spirit, one that appeals more to English education than to that of any other country, and one that comes just at the time for bidding farewell, if such must be the case, in royal fashion to the text-book whose history it sets forth.

The purpose of the author is to give a new and better translation of Euclid's elements, to recite the history of the most important definitions and propositions, to set forth the criticisms and suggested improvements that have been offered from time to time, and, incidentally in appearance but designedly in fact, to show that, as compared with the great mass of modern text-books, the Alexandrian classic stands like a mountain surrounded by foothills. The feelings of the author with respect to this supremacy of the work upon which he is writing appears in his opening quotation from De Morgan: "There never has been, and till we see it we never shall believe that there can be, a system of geometry worthy the name, which has any material departures (we do not speak of corrections or extensions or developments) from the plan laid down by Euclid." Perhaps it is a sufficient tribute to Dr. Heath's labors to say that few will be able to read his work without agreeing with him and with the illustrious predecessor whom he quotes.

A work of this nature is very welcome at this time in the development of education, and England is the natural place for its appearance. For not only has Euclid flourished on British soil as nowhere else, but English writers have left the field open by producing only text-books of Euclid rather than works showing a larger appreciation of his system. Therefore Dr. Heath sets for himself four tasks: (1) To make a new and more careful translation of the Elements, a labor that has not been attempted since Heiberg issued his scholarly Latin edition of the text, together with the Greek original, some twenty years ago; (2) to include Books VII, VIII, IX, X, and XIII, with the so-called Book XIV in an appendix, none of these having appeared in English since Williamson's treatise of 1781–88;

(3) to add such critical notes as should summarize the scientific discussion of all important definitions, postulates, and propositions, digesting such treatises as those of Schotten,[*] Veronese,[†] and Engel and Stäckel;[‡] (4) to embody also the results of the recent historical discussions of scholars like Tannery, Heiberg, Hankel, Bretschneider, Cantor, Loria, and Zeuthen.

The first volume is devoted to the Introduction, and to Books I and II of the Elements. The Introduction first discusses all that is positively known of the life of Euclid, and in this it adds nothing to our stock of knowledge, since no recent discoveries have thrown light upon the subject. It brings together in a few pages, however, the opinions of the early Greek writers and furnishes a good bibliography through which students can work. It then devotes one chapter to Euclid's other works; one to Greek commentators other than Proclus; one to Proclus and his sources; one to the text and to a discussion of the manuscripts used by Heiberg, with a critical examination of the important question of possible alterations made by Theon; one to the scholia, in which the author accepts Heiberg's conclusion that all the lemmas are spurious and are largely due to the commentary of Pappus; one to Euclid in Arabia, and one to the nature of the Elements. It is well known that our knowledge of Euclid in the West, in the middle ages, came not from any Greek manuscripts or Latin translations, but from the Arabic. In his chapter on Euclid in Arabia the author has gathered a considerable amount of recent material and has made us his debtor by setting it forth in condensed form. He shows that if the Elements was not the first it was among the first of the books translated from the Greek into the Arabic, the earliest version being that of al-Hajjāj ben Yūsuf ben Matar,[§] who made two translations. It was next translated by Ishāq ben Ḥunain and this rendering was improved by Thābit ben Qurra (died 901). The third great translation was that of Naṣīraddīn at-Ṭūsī,[‖] which was published in Rome in the Arabic in 1594, in reality a paraphrase of Euclid based on Arabic texts. From the Arabic the work got into Latin through the translation of

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[*] Inhalt und Methode des planimetrischen Unterrichts, Leipzig, 1890.

[†] Fondamenti di geometria, Padova, 1891.

[‡] Die Theorie der Parallellinien von Euklid bis Gauss, Leipzig, 1895.

[§] El-Haǧǧâǧ ben Jûsuf ben Matar, to take Suter's transliteration. He lived under Hārūn al-Rašīd and al-Māmūn, and therefore c. 786–835 A. D. He also translated the Almagest.

[‖] The Nasîr-Ed-dîn el-Ṭûsî of Suter. He was born at Ṭūs in Kurāsān in 1201 and died in 1274.

Athelard of Bath (c. 1120) and the copy from it made by Campanus (c. 1250), or possibly, as Curtze thinks, from an earlier anonymous Latin version from which both Athelard and Campanus made their so-called translations. All of these standard texts Dr. Heath has compared with the Greek versions with a view to deciding upon the actual propositions originally in the Elements as they came from Euclid's hand. As is well known, the propositions deemed essential for plane geometry numbered probably 173, the excess over the 167 of a book like Wentworth's being chiefly due to some semi-algebraic propositions of Book II. A list of the most important translations and editions follows the chapter on Euclid in Arabia.

The Introduction closes with a discussion of the nature of the Elements, probably the most important single chapter in whole work. The author shows with great clearness, and by reference to the treatise itself and to the ideas of the leading ancient commentators, exactly what are the distinctive features of Euclid's work. He refers to Proclus a statement to the effect that "in the whole of geometry there are certain leading theorems, bearing to those which follow the relation of a principle, all-pervading, and furnishing proofs of many properties. Such theorems are called by the name of *elements;* and their function may be compared to that of the letters of the alphabet in relation to language, letters being indeed called by the same name in Greek (στοιχεῖα)." He discusses the question of "Elements" prior to Euclid, by which the latter may have been influenced, and then considers the subject of "First principles," showing particularly the relation of Euclid's conception of what should be postulated to that of Aristotle and the followers of Plato. There is also in this chapter a very satisfactory treatment of the history of such topics as analysis and synthesis, and of such special forms of proof as the reductio ad absurdum. The chapter closes with a historico-philosophical discussion of the nature of definitions.

Ranking with the Introduction in point of interest is the treatment of Book I, since it is upon this book that geometry so largely depends. The history of such definitions as those of point, line, straight line, surface, plane, and angle, is here set forth with a clearness of style and a critical judgment that make the book a sine qua non to every writer upon geometry and to every one who seeks to train a teacher. The discussion of angle, for example, is a model of judicial treatment. It is not that of a man who holds a brief for Euclid, but one who recognizes that the Greek classic is not without defects and who seeks to remove any fair criticism by bettering the text. An idea of the thoroughness of the discussion may be derived from the fact that five large octavo pages are devoted to a much-condensed treatment of this particular concept. In the same spirit each definition, postulate, and axiom ("common notion") is discussed, both historically and philosophically, the weak features recognized, the strong ones pointed out, and the suggestions for improvement carefully considered. In particular the author recognizes, as everyone must, Euclid's failure in not introducing some kind of principle of continuity, say like one of those proposed by Killing, or some modification of Dedekind's postulate. The postulate of motion without deformation and the congruence postulates of Pasch, Veronese, and Hilbert, have also due recognition.

It cannot be expected that a reviewer should speak at length of the commentary, historical and mathematical, that follows. Each of the important propositions of the thirteen books is treated at more or less length, the proof and the possibilities of improvement being discussed. A spirit of fairness is everywhere apparent, and the reader lays down the third volume with the feeling that the work of a master has had masterly treatment. It will be generations before any one will again undertake this task, or will feel even tempted to do so. Riccardi* stated some years ago that no book save the Bible had been through so many editions as Euclid, and he listed over a thousand different publications of the Elements. Of the notable list now known, the editions of Heiberg and Heath will rank as the greatest for a long time to come.

The author is also deserving of special thanks for having placed an index and a glossary of Greek terms at the close of each of the three volumes. Typographically the work is up to the standard of the Cambridge University Press, and even the most critical will find little to condemn either in the subject matter or in the appearance of the work.

DAVID EUGENE SMITH.

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* Saggio di una bibliografia Euclidea, Bologna, 1887, pt. I, p. 3.

SHORTER NOTICES.

Vorlesungen über Differential- und Integral-Rechnung. Von EMANUEL CZUBER. Zweite, sorgfältig durchgesehene Auflage. II Band. Leipzig, Teubner, 1906. 8vo. viii + 532 pp.

THE present book is the second volume of a course of lectures prepared by the author primarily for students in technical schools, but with the hope that it will meet the needs of students in a "narrower" sense. The first edition of the two volumes appeared in 1898 and met with a success that called forth this second edition eight years later.

The course was prepared, as appears in the preface to the first edition, in the firm conviction that the calculus has a twofold mission to perform. On the one hand, it marks the close of the mathematical training of most students in the technical school, and should therefore contain sufficient material to give a scientific conception of technical problems, and an ability to read intelligently the rich literature in this domain. On the other hand, it is the gate through which the student who elects the study and teaching of mathematics as his life work must enter his chosen field. It must therefore contain enough of the spirit of research to give the student a mental training in, and appreciation of, modern analysis and rigor, in so far as this is possible with the material chosen for exposition.

In the present volume the student is introduced at once to the fundamental problem of the integral calculus, *i. e.*, the solution of the equation

$$\frac{dF(x)}{dx} = f(x),$$

and a clear distinction is made between the *formal* solution derived from the mean value theorem of the differential calculus and a *practical* solution.

The definite integral precedes the indefinite integral, and in the first section of Chapter I one finds its definition and usual properties together with a good definition of an integrable function leading up to the *principal theorem* of the integral calculus,

$$\int_a^b f(x)dx = F(b) - F(a).$$

The remainder of the chapter is devoted to the formulas for immediate integration and the usual methods of substitution and integration by parts.

In Chapter II is found the integration of rational functions, irrational functions, and transcendental functions. The treatment could not well be otherwise than usual. One finds, however, throughout a clearness of statement and an insistence upon the distinction between a theoretically complete solution and solutions dependent upon isolated methods.

Chapter III contains the treatment of simple and multiple definite integrals. The simple definite integral has already been defined (Chapter I) and here we find methods for its evaluation together with the mean value theorems and an excellent treatment of improper integrals, not only in the cases where the indefinite integral can be expressed by foregoing methods, but also in cases where this is not possible, provided the function to be integrated does not change sign in the interval (x_0, b) where $a \leqq x_0 < b$ and a and b are the limits of integration $(b \leqq \infty)$. In case the function does not satisfy this condition, a statement of the method to be used in testing for the existence of the integral is given and two illustrative examples worked.

The use of infinite series for evaluating integrals is preceded, as it should be, by the theorem upon the integration of infinite convergent series.

There follows a treatment of the integral as a function of one of its limits; also as a function of a parameter. It is interesting to note that the double definite integral is led up to by means of integration under the integral sign, and the consequent proof that the value of the double iterated integral * between constant limits is independent of the order of integration. Double definite integrals are carefully defined and satisfactory proofs given that they can be evaluated by means of iterated integration. Triple integrals, and multiple integrals in general, follow without unnecessary repetition of proofs. Improper double integrals are defined and illustrated by examples. The chapter closes with a section devoted to eulerian integrals and Fourier series. This section has been added to the first edition of the book and materially increases its value as an exponent of the two-fold mission stated above.

The application of the integral calculus to geometry, me-

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* Czuber uses the term "Zweifaches Integral." P. du Bois-Reymond and O. Stolz use the expression "Zweimaliges Integral."

chanics, and physics has been deferred to the fourth chapter. The student, having been furnished with the tool and an appreciation of its power, as well as its limitations, is ready to use it. The calculation of moments of inertia and centers of mass as well as Green's theorems have been added to the first edition. This, together with the addition noted above, marks practically the only change from the earlier edition of the integral part of the course. The sections on applications to mechanics and physics are especially well written. The student who goes over the book thus far will have an intelligent conception of the problems in mechanics to which the integral calculus is especially applicable and the use of Laplace's equation and Poisson's equation, as well as a knowledge of the meaning of such terms as "fields of force," "force function," "tubes of force," "strength of field," etc.

The fifth and last chapter is devoted to differential equations and covers nearly one-third of the book. The effort being to emphasize the integral calculus as an instrument to be used by the student in dealing with technical problems, this is as it should be. At the same time the finer use as a mental training in mathematical thought is not lost sight of. The chapter under consideration is a concise, if elementary, treatment of differential equations and not at all a list of isolated methods for solving the various types. The greater portion of the chapter is devoted to the consideration of ordinary differential equations and contains a section upon the calculus of variations, which is necessarily limited to the older theory and its classic problems. The part devoted to partial differential equations is pushed far enough to include the Ampère equation and its special case, the Monge equation, but with the frank acknowledgment that partial differential equations of the second order have not, as yet, been completely solved.

Illustrative examples are worked throughout the text but not much space is given to lists of problems to be solved by the student — a circumstance which would indicate, even to the casual reader, more meat than bone. The book is clearly and concisely written and lacks the heaviness which characterizes some German texts. It is furnished with a good index of subjects, as well as a register of names, and is singularly free from errors. Numerous historical notes and references are given, which enhance the value of the book to the careful student.

It is safe to state that the student who has mastered the contents of both volumes will be in a position to read technical literature with intelligence, and it is equally safe to say that no student can consider himself equipped to work in any corner of the mathematical field without an equivalent body of knowledge ready at hand, or without the spirit of careful inquiry which pervades Professor Czuber's two volumes.

L. WAYLAND DOWLING.

Traité de Mathématiques générales à l' Usage des Chimistes, Physiciens, Ingenieurs, et des Élèves des Facultés des Sciences. Par E. FABRY. Avec une preface de G. DARBOUX. Paris, Hermann, 1909, 8vo. 10 + 440 pages.

DURING the last few years the requirements for the "diplôme de licencié ès sciences" have undergone considerable change. The tendency has been to imitate somewhat the German custom and allow the student greater freedom in the choice of subjects taken for this diploma. Under the present system each faculty issues certificates for the work taken in that faculty and a student who has obtained three of these certificates, whether at the same session or at different sessions, has the right to a diploma. Previously there were only three modes of obtaining this diploma, but under the new regulations Darboux tells us that at Paris alone it can be obtained in 1771 different ways.

Among the certificates most frequently issued those of general mathematics are certainly included. This course in general mathematics forms a sort of transition between the mathematics of the Lycée and the advanced mathematics of the University. Such a course has always been necessary for the student of mechanics and physics and is fast becoming necessary for the student of chemistry. It was to meet this new demand for a course in general mathematics that Professor Fabry undertook the difficult task of writing a book that should meet the needs of the engineer, physicist, and chemist and at the same time serve as a preliminary training for those who expect to pursue the study of mathematics for its own sake. Such a book must combine rigor with simplicity and in this lies the difficulty of the task. It seems that the author has succeeded remarkably well in combining these two things. To be sure at times the subject becomes quite abstract, but the student of applied science can omit these parts without in the least marring the course for him.

The book is divided into four parts: Algebra, Analytical Geometry, Analysis, Mechanics.

Part I (128 pages) starts with a discussion of incommensurables, limits, and continuity, leading to the idea of the maximum of a function $f(x)$ which remains finite between a and b. In the first 46 pages is given all that the student will need of the binomial theorem, determinants, series, and exponential functions. The subject matter of these introductory chapters is remarkably well chosen and only those things are given which are really needed for further developments. In the chapter on exponential functions and logarithms the number e is defined by the series

$$e = 1 + \frac{1}{1} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots$$

and no attempt is made to show that $(1 + 1/n)^n$ approaches e as n becomes infinite. The function e^x is then defined and the convergence of the series discussed.

Derivatives are defined and discussed in Chapter VI, page 46, and thus early in the work the student has this tool at his disposal. It was rather curious to note that the symbols dy, dx are used to denote increment instead of the old familiar ones Δy, Δx. The formulas of differentiation and Rolle's theorem are disposed of in 6 pages.

The derivation of Taylor's series in its finite form is rather interesting. Given the function $f(x)$ which possesses the first $n + 1$ derivatives, the following expression is written down

$$F(x) = f(b) - f(x) - (b - x)f'(x) - \frac{(b - x)^2}{\underline{|2}} f''(x) - \cdots$$
$$- \frac{(b - x)^n}{\underline{|n}} f^{(n)}(x) - A(b - x)^p,$$

where p is a positive integer and

$$A = \frac{1}{(b - a)^p}\left[f(b) - f(a) - (b - a)f'(a) \right.$$
$$\left. - \frac{(b - a)^2}{\underline{|2}} f''(a) - \cdots - \frac{(b - a)^n}{\underline{|n}} f^{(n)}(a) \right].$$

Now since $F(a) = 0$, $F(b) = 0$, and $F'(x)$ exists for all values of x between a and b, we have by Rolle's theorem

$$F'(x) = -\frac{(b-x)^n}{\underline{|n}} f^{(n+1)}(x) + Ap(b-x)^{p-1} = 0$$

for some value x_0 of x between a and b; and as $b - x_0 \neq 0$

$$A = \frac{(b-x_0)^{n-p+1}}{\underline{|n}} f^{(n+1)}(x_0).$$

Equating this to the value of A above and making the substitutions

$$a = x, \quad b = x + h, \quad x_0 = x + \theta h \qquad (h < 1),$$

we have Taylor's series in its finite form. The remainder is of the form

$$\frac{h^{n+1}}{\underline{|n}} \cdot \frac{(1-\theta)^{n-p+1}}{p} f^{(n+1)}(x + \theta h).$$

If $p = n + 1$, we at once have Lagrange's form of the remainder. If $p = 1$, we obtain Cauchy's form of the remainder.

Chapter VII is devoted to applications of differentiation; but as no geometry has yet been developed, about the only applications made are to maxima and minima and the evaluation of indeterminate forms. Chapter VIII treats of development in series. Uniform convergence is defined and criteria are given for the differentiation of series term by term. The expansion of a function by means of Taylor's series is discussed and the expansion is applied to a few particular functions. In Chapter IX functions of several variables are taken up. It is remarkable that the subject should be developed thus far (80 pages) before the first figure appears. It would seem extremely difficult for the beginner to grasp the full meaning of much of the abstract mathematics without some sort of a geometrical picture. It is rather unfortunate that the whole subject of maxima and minima should be developed without the use of a single graph.

Chapter X is a short introduction to the study of imaginaries, and the remainder of Part I is devoted to algebraic equations, rational fractions, elimination, etc.

Part II, Analytical geometry (145 pages) contains the elements of both plane and solid analytical geometry besides most of the applications of differential calculus to geometry contained in elementary treatises on the calculus. Multiple points, asymptotes, unicursal curves, curvature, torsion, osculating plane, ruled and developable surfaces are among the topics discussed. The chapters on the conic and quadric are short and yet contain the important properties of these curves and surfaces. The conics are discussed purely from the equation and not as a locus problem. One chapter however is devoted to geometric loci, in which the general idea of a locus is developed and such curves as the strophoid, cissoid, and cycloid are discussed as examples in loci. The idea of a multiple point is introduced after 28 pages. It seems as if it might have been better to delay this until the student had a little more familiarity with curves in general.

Part III, Analysis (85 pages), begins with a discussion of infinitesimals and differentials. Here the notation is changed, Δy now being used to represent increment and dy to represent differential. In the first chapter the handling of differentials is pretty thoroughly drilled into the student. The expression for radius of curvature which has already been derived in terms of derivatives is now expressed in terms of differentials in both rectangular and polar coordinates. The definite integral is then defined and it is shown that the limit of $\Sigma m_i \delta_i$ is the same no matter how the interval be divided. It is then shown that the limits of $\Sigma m_i \delta_i$ and $\Sigma M_i \delta_i$ (M_i and m_i being the maximum and minimum of $f(x)$ in the interval δ) are the same, if $f(x)$ is a continuous function. This is rather abstract mathematics for students of applied science, but the author has a scientific conscience to satisfy, and besides it will be useful to those who expect to continue the study of pure mathematics. In a few pages the ordinary rules of integration are developed. The evaluation of the definite integral is then taken up and the criteria that the integrals

$$\int_a^b f(x)dx,$$

where $f(x)$ becomes infinite between a and b, and

$$\int_a^\infty f(x)dx$$

should exist, are given. Evaluation of a definite integral by expanding the integrand and integrating term by term is shown by means of a single illustrative example. Multiple integrals are defined and the method of evaluating them is given without a rigorous demonstration, and applications are made to area, volume and surfaces of solids. In Chapters VI and VII line and surface integrals are discussed and Stokes's and Green's theorems are derived.

The last twenty pages of Part III are devoted to differential equations. Here again the author shows his ability to present the essentials of a subject clearly and in a small space. The linear equation, singular solution, systems of equations and partial differential equations, all come in for their share in these few pages. The student of applied science will find here about all the differential equations ordinarily necessary for his purpose.

Part IV, Mechanics (60 pages). In these few pages the student will find an introduction to theoretical mechanics. The subject is opened by a discussion of velocity and acceleration and the representation of the acceleration by means of the hodograph. The components of the acceleration along the tangent and principal normal of the curve of motion are then formed, which leads to the equation

$$\gamma^2 = \left(\frac{dv}{dt}\right)^2 + \frac{v^4}{\rho^2} \quad [\rho \text{ being the radius of curvature}].$$

By means of this equation various kinds of motion are discussed. Motion expressed in terms of polar coordinates is then taken up and the expression for the acceleration is derived. The instantaneous center of curvature is introduced both for plane and solid figures and the parametric equations of its path are obtained. This first chapter then closes with a good discussion of relative motion. The chapter contains only eight pages and from the amount of material which it contains some idea can be formed of the amount of mechanics which the author compressed into sixty pages. The following chapters are entitled respectively, Dynamics of a free point; Dynamics of a point not free to move; Center of gravity; Dynamics of systems. The whole treatment is concise and to the point and mathematical in all its details.

At the end of each chapter there are a few well chosen examples. The book closes with a table of formulas which contains, besides the principal results derived in the text, also all the most important trigonometric and geometric formulas useful to the student.

C. L. E. MOORE.

Auslese aus meiner Unterrichts- und Vorlesungspraxis. By HERMANN SCHUBERT. 3 volumes. G. J. Göschen, Leipzig, 1905, 1906.

IN these little books the veteran teacher and editor treats a great variety of topics more or less closely connected with the mathematical subjects studied in the German gymnasium. Originality and elegance of presentation, even in the case of the most hackneyed topics, make the lectures of interest and value. The keynote is struck in the following sentence from the author's preface:

"Denn die mathematische Didaktik sollte sich nicht auf oft erfolglose Verbesserungsvorschläge bezüglich der Verteilung und der Ausdehnung des Lernstoffs beschränken, sondern sollte umgekehrt die Pflicht fühlen und erfüllen, den zu bewältigenden Lernstoff so einfach und zugänglich zu gestalten, dass auch der minder begabte Schüler in der nun einmal von oben herab vorgeschriebenen Zeit mehr lernt und begreift, als es bisher der Fall war."

Two of the chapters deal with the calculation of logarithms, the first intended for students of the Untersekunda, the second for the Prima. The first treatment presupposes merely a knowledge of the expansions of $(a-b)(a+b)$, $(a-b)^2$, $(a-b)^3$, $(a-b)^4$, and the theorems concerning the logarithms of products and quotients, and is based on the inequality

$$2 \log x - \log (x-1) - \log (x+1) > 0.$$

The second treatment presupposes the general binomial theorem and is based on the so-called "Tripelformel" originally presented in the author's pamphlet "Elementare Berechnung der Logarithmen" (Göschen, 1903). With these elementary means, without any reference to infinite processes, the logarithms of the prime numbers less than a hundred are worked out to eight decimal places.

Many of the lectures are devoted to topics from the theory of numbers. A section of almost a hundred pages shows how the leading theorems on congruences, including quadratic remainders and the Pell equation, may be easily and rapidly obtained by starting out with the discussion of continued fractions.

The applications of number theory in elementary geometry are treated very fully. Here many of the results, not merely the presentation, are due to the author. The discussion of heronian triangles is a model of elegance. Such a triangle is one in which the area and the sides are expressible in integers. The author's method is based on the fact that the tangent of half each angle in such a triangle must be rational. Angles with this property are termed heronian. The theorem on page 13 of volume 2 should evidently be corrected so as to read: Every linear homogeneous function, with integral coefficients, of any number of heronian angles is itself an heronian angle.

The art of manufacturing problems in such a way that the solution as well as the data shall be integral is not cultivated seriously in this country.* The author takes up questions of this sort in many complicated cases: quadrilaterals with rational sides, diagonals, and area; tetrahedrons with rational edges, faces, and volume; and finally, in the last chapter of the third volume, spherical triangles with both sides and angles heronian (it is here shown that if four of the parts are heronian, the remaining two must also be heronian).

The proof of Euler's theorem on polyhedrons given in chapter 8 of the first volume is quite novel: the result is obtained by counting, in two different ways, the number of conditions which determine a polyhedron, and equating the results.

A number of the topics are taken from physics. We mention only the somewhat polemical discussion of the absolute system of units; and an elementary treatment of the pendulum, in which the period of oscillation is determined with great accuracy. The formula

$$\pi\sqrt{\frac{l}{g}}\left(1+\frac{1}{4}\sin^2\frac{\alpha}{2}\right)<t<\pi\sqrt{\frac{l}{g}}\sec\frac{\alpha}{2},$$

where α denotes the angle of oscillation, was originally published in the *Naturwissenschaftliche Wochenschrift* for 1896, but has not attracted the attention it merits.

E. KASNER.

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*Students of elementary geometry usually get the impression that if two solids agree in both surface and volume they are congruent or at least symmetric. It is of course easy to give examples showing that this is not true, even in the simple case of rectangular parallelepipeds. An example in which the dimensions are all integral would be of interest. The problem is to find integral solutions (the smaller the better) of the pair of equations

$$xyz=x'y'z',\quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{x'}+\frac{1}{y'}+\frac{1}{z'}.$$

Il Passato ed il Presente delli Principali Teorie geometriche. Terza Edizione accresciuta di uno Sguardo allo Sviluppo della Geometria in quest'ultimo Decennio. By GINO LORIA. Turin, C. Clausen, 1907. xxiii + 475 pp.

LORIA's well known history of modern geometrical theories originally appeared serially in the *Memorie della R. Accademia delle Scienze di Torino.* Ten years later, in 1896, a second edition was published in book form, and a German translation by Sturm made the work more accessible. The present third edition, a volume of almost five hundred pages, is especially valuable on account of the appended sketch of the development of geometry during the last decade, which occupies over a hundred pages.

The first chapter of the book gives a rapid survey of the origin and development of geometry up to the middle of the nineteenth century. (An English translation by Halsted appeared in the *Monist* for October, 1902.) Then follow separate chapters on the principal modern theories: plane algebraic curves, algebraic surfaces, twisted curves, differential geometry, analysis situs and configurations, line geometry, correspondences and transformations, enumerative geometry, non-euclidean geometry, hyperspaces. The final chapter is devoted to miscellaneous topics ranging from kinematical and constructive geometry to quaternions and vector analysis.

The appendix is divided into sections numbered and named in accordance with the chapters enumerated above. After a few introductory paragraphs on each topic, most of the space is devoted to the bibliographies. In some cases these extend up to 1907, in others only to 1904. Of course no attempt is made to give complete lists. In the entire work, estimating from the excellent index, references are given to perhaps 5,000 memoirs by 1,500 authors. Considerable space is saved by giving in each case only the number of the volume and the date of publication, without the page numbers.

In a bibliographical work of this kind more energy might have been devoted to the proofreading; the misspelling of proper names is somewhat annoying. Fortunately hardly any of the errors are likely to mislead the reader.

It is interesting to compare the material gathered under the title non-euclidean geometry in the appendix with the corresponding chapter devoted to the literature up to 1897. The latter deals exclusively with the elliptic and hyperbolic geometries; the former with general axiomatic foundations. The activity of investigation in hyperspace geometry, especially in Italy, is put in concrete evidence by the fact that the section devoted to this subject is the longest in the appendix.

E. KASNER.

Führer durch die mathematische Literatur mit besonderer Berücksichtigung der historisch wichtigen Schriften. Von FELIX MÜLLER. Leipzig und Berlin, Teubner, 1909. x + 252 pp.

IN the preface we are told that the present work aims to be a reliable guide through the mathematical literature for all those who may seek self-instruction in any domain of the science. It seems especially suited for those students who desire to complete their lecture notes by means of references to the literature and the history of the subject. In view of the size and the scope of the work it is evident that the specialist will, in general, make little use of it along the line of his specialty, but he will find in it much general information in a convenient form. It has been made generally and permanently accessible by its appearance in the *Abhandlungen zur Geschichte der mathematischen Wissenschaften*, Heft XXVII.

In general, each chapter begins with a brief statement of the main features of the subject to which the chapter is devoted and some historical notes relating to the origin of the subject. This is followed, in order, by a selection of classic works, newer textbooks, fundamental memoirs, and articles devoted to special parts of the subject. Great care has been exercised as regards the exact titles and references, but the number of these is so large as to make errors almost unavoidable. Some of them have been corrected on the five pages of Nachträge und Verbesserungen. Among those which have escaped we may mention W. J. Stringham on page 219 in place of W. I. Stringham, J. J. Hutchinson on pages 124 and 242 in place of J. I. Hutchinson, two incorrect numbers (7, 518) following Blichfeldt on page 236, and two (205, 218) following Halsted on page 241. The latter numbers should have been entered under Halphen.

The first forty-seven pages are devoted to the history of mathematics, biography, collected works, periodicals, bibliography, and encyclopedias. For many readers this will doubtless prove the most valuable part of the volume, since it contains a large amount of information which cannot be easily gathered from other sources. The information in regard to the publications of various academies and societies, with details as to the dates of the most important serials, should prove to be of great value, not only to the student of the mathematical literature but also to those in charge of securing complete sets of important publications for libraries. The rapidly growing interest in the history of mathematics has doubtless led many professional mathematicians to wish for a reliable guide through the subjects treated in this part.

The second part covers eighty-four pages and is devoted to the following subjects: philosophy, pedagogy, algebra, arithmetic, analysis, and function theory. Although nearly all the space is used for references yet the extent of the literature on these subjects is so great that only a very small part of the total number of important references could be given. It can scarcely be expected that all would agree that the selections were the best that could have been made, as so much depends upon the point of view. American readers will especially miss such references as Dickson's Linear Groups, 1901; Bolza's Lectures on the Calculus of Variation, 1904; Pierpont's Theory of Functions, 1905; Bôcher's Introduction to Higher Algebra, 1907. Notwithstanding some unfortunate omissions this part contains a great deal of information in a convenient form, especially for beginners in these subjects.

The third and largest part of the volume under review is devoted to geometry, which is considered in sixteen chapters bearing the following headings: Foundations of geometry, elementary plane geometry, trigonometry, continuity, solid geometry, descriptive geometry, analytic geometry, synthetic geometry, infinitesimal geometry, higher plane curves, higher solid geometry, enumerative geometry, line geometry, transformations, geometry of more than three dimensions, kinematic geometry. Although a little more than one-third of the volume is devoted to this part yet some very important memoirs and works are not mentioned in the references. Among these is Kasner's excellent report on the present problems of geometry, which appeared in the BULLETIN in 1905.

The work closes with a valuable table of contents and a list of names quoted, with references to the pages. The preparation of a reliable guide through the mathematical literature clearly calls for very high attainments on the part of its author. By his long connection with the *Fortschritte der Mathematik* and his many valuable earlier publications, the author of the present work has won the confidence of the mathematical public, and the usefulness of the present volume will doubtless increase this confidence. It is hoped that it may also inspire others to render such scholarly services even if their accomplishment demands a vast amount of time, care, and patience. The more our science grows the more need there will be for such works, and conversely such works contribute materially towards stimulating real growth.

G. A. MILLER.

Ueber das Wesen der Mathematik. Rede gehalten am 11. März 1908 in der öffentlichen Sitzung der K. Bayerischen Akademie der Wissenschaften von DR. A. VOSS, O. Professor der Mathematik in München. Erweitert und mit Anmerkungen versehen. Leipzig und Berlin, B. G. Teubner, 1908. 98 pp.

ADDRESSING a mixed assembly of scientists, Voss endeavors to answer the questions: What is the nature (Wesen) of mathematics? How does it happen that mathematics is the only science which presents truth in apodictic form? What has the past century accomplished toward the elucidation of the inner structure of mathematics?

Though not answering these difficult questions with the precision some may desire, the address is extremely able and instructive. The annotations occupy about half the space of the book and contain numerous valuable references to the literature of the subject. The author passes in rapid historical review the fundamental concepts of variables, functional dependence, and limits. He points out the triumphs of mathematics during the eighteenth century, a period which culminated with Laplace. This great savant made the well-known utterance, now admitted to be a gross exaggeration: An intelligence to whom at a given moment were presented the conditions of the entire material world would be able by mathematical analysis to survey the entire past and future of the world. In considering the various attempts to define mathematics, Voss passes from the antiquated "science of quantity" definition to the more recent ones based on the consideration of the logical steps involved, namely the definitions of B. Peirce, E. Papperitz, G. Itelson, B. Kempe, M. Bôcher, H. B. Russell, and L. Couturat. Geometry and mechanics belong to applied mathematics. Pure mathematics has come to be considered the science of number (die Wissenschaft von den Zahlen). There follows a sketch of the historical development of the number concept. It is pointed out that the axiom of Eudoxus and Archimedes does not hold for complex numbers. If $a + bi \gtrless c + di$, when $a \gtrless c$, or when $a = c$ and $b \gtrless d$, then not only is $b^2 i$ less than $c^2 + di$, but $nb^2 i$ is less also, no matter how large a positive integer n may be. Nor does the axiom hold for types of real numbers introduced by Hilbert in his Grundlagen. Voss discusses the nineteenth century researches on the concept of a function of a complex variable, on differential equations, on Cauchy's and Riemann's definitions of an integral, the recent theory of manifoldness and the paradoxes which still encumber it. Toward the close he touches upon geometry, makes war upon Wundt for his critique of non-euclidean geometry, and speaks of the work of Peano and Hilbert on the foundations of geometry. Lastly he raises the question, is mathematics after all not a huge tautology? If everything rests upon number and its assumed laws of combination, then everything is included in the logical unfolding of the postulates, and a powerful intellect should be able to view all consequences at a glance. Poincaré answers the question in the negative, for mathematical reasoning involves a creative power, displayed in the act of the so-called mathematical induction from n to $n + 1$. Fr. Meyer answers that, from even a small number of axioms, one may draw an unlimited number of deductions and the whole science reduces itself to a gigantic game of chess. Voss himself likewise answers the question in the negative, but considers both explanations inadequate in the light of history. In the ability of the human mind to make new experiences, to draw from them general ideas and subject these to mathematical analysis, to numerical treatment, we must seek the steady growth of the science. Thus, the idea of continuity gave rise to irrational number, the idea of motion to the concept of a limit.

We have noticed only a few historical errors. Logarithms were invented in 1614, not 1620 (page 11); the Diophantine symbol of subtraction is given incorrectly (page 11); the principle of local value in writing numbers is ascribed to the Hindus (page 10), but was known earlier by the Babylonians, at least in part. In referring to Babylonian mathematics, Voss fails to mention the important researches of Hilprecht, of the University of Pennsylvania.

On the authority of Moritz Cantor,* Voss ascribes the introduction of the inference by mathematical induction to Pascal (before 1654) and its independent re-introduction to Jacob Bernoulli (1680). This important process is called by Max Simon † the "Bernoullian" or the "Kästnerian" inference. Charles S. Peirce ascribes it to Fermat and calls it the "Fermatian" inference.‡

In view of these conflicting historical statements, it is worth while to inquire more minutely into the origin of the process of mathematical induction. The claim for Kästner hardly needs consideration; he and other writers used it in the second half of the eighteenth century. Nor can priority be established for Jacob Bernoulli. The claim for Pascal rests upon passages in his Traité du triangle arithmétique, particularly the one relating to a certain proportion: § "Quoique cette proposition ait une infinité de cas, j'en donnerai une démonstration bien courte, en supposant deux lemmes." The first lemma says that the proportion is seen to be true for numbers on the second diagonal ("base"); the second lemma asserts, "que si cette proportion se trouve dans une base quelconque, elle se trouvera nécessairement dans la base suivante. D'où il se voit qu'elle est nécessairement dans toutes les bases : car elle est dans la seconde base par le premier lemme ; donc par le second elle est dans la troisième base, donc dans la quatrième, et à l'infini." This process is precisely what is now designated by mathematical induction.

Charles S. Peirce does not cite, in substantiation of his claim, any particular passage from the writings of Fermat, nor does he make any specific reference to them. Peirce describes mathematical induction in the manner current in modern texts and ascribes it to Fermat. We have not been able to find in Fermat's works the process of mathematical induction in its purity. In the famous Fermatian document, discovered among the papers of Huygens in 1879, and entitled "Relation des découvertes en la science des nombres," Fermat describes a method which he calls "la descente infinie ou indéfinie." He says that it was particularly applicable in proving the impossibility of certain rela-

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* Vorlesungen über Geschichte der Mathematik, vol. 2, 2te Aufl., Leipzig, 1900, p. 749.

† Methodik d. elem. Arithmetik, Leipzig und Berlin, 1906, p. 13.

‡ Century Dictionary, Art. "Induction;" *The Monist*, vol. 2 (1892), pp. 539, 545.

§ Oeuvres complètes de Blaise Pascal, vol. 3, Paris, 1866, p. 248 (*conséquence* XII).

tions, but that he finally succeeded in using the method also in proving affirmative statements. "Si un nombre premier pris à discretion qui surpasse de l'unité un multiple de 4 n'est point composé de deux quarrez, il y aura un nombre premier de mesme nature moindre que le donné; et ensuite un troisième encore moindre, etc., en descendant a l'infini jusques à ce que vous arriviez au nombre 5, qui est le moindre de tous ceux de cette nature, lequel il s'en suivroit n'estre pas composé de deux quarrez, ce qu'il est pourtant d'ou on doit inferer par la deduction à l'impossible que tous ceux de cette nature sont par conséquent composez de 2 quarrez." This famous method, known to Fermat probably as early as 1636 or 1637, contains a recurrent mode of inference, embodying in one step an indefinitely large number of syllogisms. But it is not mathematical induction in its purity; it is such a process applied to a descending order of progression, and appearing as the superstructure to a reductio ad absurdum argument. Moreover, Fermat does not descend in regular progression from n to $n - 1$, $n - 2$, etc., but leaps irregularly over usually several integers from n to $n - n_1$, $n - n_1 - n_2$, etc. Neither Moritz Cantor * nor H. G. Zeuthen † calls this mathematical induction or "vollständige Induktion."

Even if this term be used in a sense broad enough to include the mode of inference described by Fermat, should it be called the "Fermatian inference"? We think not. In the first place his Relation did not become general property of mathematicians until 1879. Fermat's leaning was toward keeping his methods secret. He influenced mathematicians in the theory of numbers by the *results* he reached, rather than by the methods he made known. In the second place, the process in question was used 300 years earlier by Campanus in his edition of Euclid (1260). Campanus proves the irrationality of the golden section. His argument is reproduced by Genocchi and Cantor in algebraic notation thus: ‡ If $(x_1 + x_2) : x_1 = x_1 : x_2$, then also $x_1 : x_2 = x_2 : (x_1 - x_2)$ or, putting $x_1 - x_2 = x_3$, where $x_1 > x_2$, we have $(x_2 + x_3) : x_2 = x_2 : x_3$, where $x_2 > x_3$. In the same way we get $(x_3 + x_4) : x_3 = x_3 : x_4$, where $x_4 = x_2 - x_3 < x_3$, and so on indefinitely. Since there is only a finite number of positive integers less than $x_1 + x_2$, the first proportion is incorrect; x_1 and x_2 can not be integers; q. e. d.

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* Vorlesungen über Geschichte der Mathematik, vol. 2, 2te Aufl., Leipzig, 1900, pp. 778–780.

† Geschichte der Mathematik im XVI. und XVII. Jahrhundert, deutsche Ausg. v. Raphael Meyer, Leipzig, 1903, pp. 157–162.

‡ Cantor's Vorlesungen, vol. 2, 2. Aufl. 1900, p. 105.

The theory of numbers received much attention from the Hindus and Greeks, and it would be surprising if the recurrent mode of inference were not found among them. It does not appear, however, strictly in the form of the modern mathematical induction and free from entanglement with other processes. The "cyclic method" of Bhaskara, for solving $ay^2 + s = x^2$, led to results enabling him from one integral (or rational) solution, $x = p_1$, $y = q_1$, of the more special equation $ay^2 + 1 = x^2$, to get a second solution, $p_2 = p_1^2 + aq_1^2$, $q_2 = 2p_1q_1$ and, by repetition, an infinite series of solutions of $ay^2 + 1 = x^2$.[*]

Perhaps less widely known is the rule given by Theon of Smyrna (about 130 A. D.) and proved by Proclus (410–485 A. D.) for finding numbers representing the sides and diagonals of squares, *i. e.* satisfying $d^2 = 2s^2$. If d_1 and s_1 is a solution, so is $d_2 = 2s_1 + d_1$, $s_2 = s_1 + d_1$. But $s_1 = 1$ and $d_1 = \sqrt{2}$ is a solution. Consequently, we get an indefinitely large number of solutions $d_2, s_2, d_3, s_3, \cdots$ Similar processes are given by Proclus for $x^2 - 2y^2 = 1$. It is believed that these problems were studied by the Pythagoreans and by peoples of still earlier times.[†] In these Greek and Hindu writings the recurrent process is used for finding one solution from another solution, rather than to show the generality of some particular formula or theorem. It is easy to see how some slight change in the mode of presentation or in the point of view would yield the modern mathematical induction. Thus Proclus was very close to the establishment of the formulas $d_{n+1} = 2s_n + d_n$, $s_{n+1} = s_n + d_n$, for any value of n.

All in all it seems best to restrict the term "mathematical induction" strictly to the modern definition of the term, and to ascribe it to Pascal. The recurrent modes of inference of the Hindus and Greeks are more nearly the modern process of mathematical induction than is the mode of inference used by Fermat.

FLORIAN CAJORI.

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[*] H. Hankel, Geschichte d. Mathematik, Leipzig, 1874, p. 200.

[†] Fr. Hultsch, "Die Pythagoreischen Reihen, etc.," in *Bibliotheca Mathematica*, 3d ser., vol. 1, 1900, pp. 8–11; H. Konen, Geschichte d. Gleichung $t^2 - Du^2 = 1$, Leipzig, 1901, pp. 2–11.

NOTES.

A NUMBER of copies of the first five volumes of the *Transactions* of the Society, the covers of which have become soiled with dust, but which are otherwise in good condition, are offered to members of the Society at $2.50 per volume. Orders should be sent to the Secretary.

THE April number (volume 10, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "General theory of modular invariants," by L. E. DICKSON; "Beiträge zur Theorie der Gruppen linear homogener Substitutionen," by I. SCHUR; "Projective differential geometry of curved surfaces (fourth memoir)," by E. J. WILCZYNSKI; "Natural families of trajectories; conservative fields of force," by E. KASNER; "Plane fields of force whose trajectories are invariant under a projective group," by G. W. HARTWELL; "On the order of primitive groups," by W. A. MANNING; "Existence and oscillation theorem for a certain boundary value problem," by G. D. BIRKHOFF; "On the regions of convergence of power series which represent two dimensional harmonic functions," by M. BÔCHER.

THE April number (volume 31, number 2) of the *American Journal of Mathematics* contains: "Rational reduction of a pair of binary quadratic forms; their modular invariants," by L. E. DICKSON; "Surfaces and congruences derived from the cubic variety having a double line in four-dimensional space," by V. SNYDER; "Finite groups which may be defined by two operators satisfying two conditions," by G. A. MILLER; "Symmetric binary forms and involutions," by A. B. COBLE.

THE April number (volume 10, number 3) of the *Annals of Mathematics* contains: "A method of investigating numbers of the form $6^{\kappa}s \pm 1$," by L. L. DINES; "The solution of algebraic equations by partial differential equations," by H. A. SAYRE; "The in- and circumscribed quadrilateral," by W. E. BYERLY; "Applications of probabilities to mechanics," by E. B. WILSON.

AT the meeting of the London mathematical society held on March 11 the following papers were read: By J. LARMOR, "The kinetic image of a convected electric system in a conducting plane sheet"; by G. H. HARDY, "On an integral equation"; by H. BATEMAN, "Transformation of electrodynamic equations and the laws of motion"; by E. CUNNINGHAM, "Transformation of electrodynamic equations of moving bodies."

THE Smith prizes of Cambridge University for the year 1909 were awarded to N. W. TURNBULL (Trinity), for his essay: "The irreducible concomitants of two quadrics in n variables," and to G. N. WATSON (Trinity), for his essay: "The solution of the linear homogeneous difference equation of the second order and its application to the theory of linear differential equations of the Fuchsian type."

THE following members of the International Commission on the Teaching of Mathematics, established by the International Congress of Mathematicians held at Rome in April, 1908, have thus far been appointed: Austria, E. Czuber, R. Suppantschitsch, W. Wirtinger; Denmark, P. Heegaard; France, P. Appell, C. Bourlet, C. A. Laisant; Germany, F. Klein, P. Staeckel, P. Treutlein; Greece, K. Stephanos; Great Britain, Sir G. Greenhill; Holland, J. Cardinaal; Hungary, E. Beke, G. Rados; Italy, G. Castelnuovo, F. Enriques, G. Vailati; Portugal, G. Teixeira; Russia, N. J. Sonin, B. M. Kojalovič, M. Vogt; Spain, Z. G. de Galdeano; Switzerland, H. Fehr, C. F. Geiser; United States, W. F. Osgood, D. E. Smith, J. W. A. Young. The United States section has organized and elected Professor D. E. Smith chairman, and a preliminary report has been issued. The work in this country will be carried on by means of a number of committees and subcommittees soon to be appointed.

THE publishing house of Ginn and Company, of New York and Boston, announce a treatise on differential geometry by Professor L. P. EISENHART, of Princeton University, now in press and to appear during the early summer.

THE firm of Martin Schilling, manufacturer of mathematical models and apparatus announces that following the death of Mr. Schilling its address is changed to Leipzig, Kantstrasse 12. Its affairs will be conducted by a competent scientific director, as heretofore. A large number of new models for use in descriptive geometry, theory of curves and surfaces, theory of functions, algebra, and mechanics are being prepared, most of which will appear this spring.

WITH the April number of the *American Mathematical Monthly*, Professor L. E. Dickson retires from the board of editors; the staff now consists of Professors B. F. FINKEL, G. A. MILLER and H. E. SLAUGHT. The place of publication has been transferred to Chicago.

THE following courses in mathematics are offered during the summer semester of 1909:

UNIVERSITY OF BERLIN. — By Professor H. A. SCHWARZ: Problems in maxima and minima treated by methods of elementary geometry, two hours; Integral calculus, four hours; with exercises, two hours; Theory of analytic functions, I, four hours; Colloquium, two hours; Seminar, two hours. — By Professor G. FROBENIUS: Theory of algebraic equations, II, four hours; Seminar, two hours. — By Professor F. SCHOTTKY: Theory of elliptic functions, four hours; Theory of potential, two hours; Seminar, two hours. — By Professor G. HETTNER: Theory of probabilities and errors of observation, two hours. — By Professor J. KNOBLAUCH: Analytic geometry, four hours; Theory of surfaces and space curves, four hours; Congruences, one hour. — By Professor E. LEHMANN-FILHÈS: Differential calculus with exercises, four hours. — By Dr. I. SCHUR: Introduction to the theory of ordinary differential equations, four hours; Theory of algebraic numbers and ideals, four hours.

UNIVERSITY OF BONN. — By Professor E. STUDY: Applications of the calculus to geometry, four hours; Introduction to the theory of invariants, one hour. — By Professor E. LONDON: Differential and integral calculus, four hours; Axonometry and perspective, two hours; Seminar, two hours. — By Professor G. KOWALEWSKI: Theory and applications of determinants, three hours; Differential equations, four hours; Method of an infinite number of variables, one hour. — By Professor C. CARATHÉODORY: Calculus of variations, four hours; Seminar, two hours. — By Dr. F. HESSENBERG: Theory of algebraic equations, four hours.

UNIVERSITY OF LEIPZIG. — By Professor C. NEUMANN: Theory of potential and spherical harmonics, four hours; Seminar, two hours. — By Professor K. ROHN: Projective geometry, two hours; Plane analytic geometry, four hours; Seminar, two hours. — By Professor O. HÖLDER: General theory of functions of a complex variable, four hours; Higher algebra, three hours; Seminar, two hours. — By Professor F. HAUSDORFF:

Theory of numbers, three hours; Theory of determinants, two hours. — By Professor H. LIEBMANN: Differential equations with exercises, six hours.

THE following courses in mathematics are announced for the year 1909–1910:

UNIVERSITY OF CHICAGO (summer quarter, June 21 to September 3, 1909). — By Professor E. H. MOORE: General analysis, four hours; Synthetic geometry, four hours; Graphical algebra, four hours, all first term. — By Professor L. E. DICKSON: Theory of numbers, four hours; Solid analytic geometry, five hours. — By Professor J. W. A. YOUNG: Critical review of secondary mathematics, four hours; Limits and series, four hours. — By Professor J. B. SHAW: Elliptic integrals and Fourier series, four hours; College algebra, five hours; Trigonometry, five hours. — By Professor O. D. KELLOGG: Theory of functions of a complex variable, four hours; Integral calculus, five hours; Analytic geometry, five hours. — By Dr. A. C. LUNN: Vector analysis, four hours; Differential calculus, five hours. — By Professor K. LAVES: Analytic mechanics, four hours. — By Dr. W. D. MACMILLAN: Introduction to celestial mechanics, five hours.

COLUMBIA UNIVERSITY. — By Professor T. S. FISKE: Advanced calculus, introduction to the theory of functions of a real variable, three hours; Theory of functions of a complex variable, three hours. — By Professor F. N. COLE: Theory of groups, three hours; Theory of invariants, three hours. — By Professor JAMES MACLAY: Application of the calculus to the theory of surfaces, three hours. — By Professor D. E. SMITH: History of mathematics, two hours. — By Professor C. J. KEYSER: General theory of assemblages, three hours; Modern theories in geometry, three hours. — By Professor H. B. MITCHELL: Geometrical analysis, three hours. — By Professor EDWARD KASNER: Geometry of dynamical systems, two hours; Contact transformations and applications, two hours. — By Professor G. H. LING: Theory of numbers, three hours; Modern higher algebra, three hours.

CORNELL UNIVERSITY. — By Professor J. MCMAHON: Vector analysis, two hours; Applications of vector fields, two hours; Theory of probabilities, two hours. — By Professor J. H. TANNER: Teachers' course (assisted by Dr. F. W. OWENS), three hours. — By Professor J. I. HUTCHINSON : Differential geometry, three hours (first term). — By Professor V. SNYDER : Algebraic curves, three hours. — By Professor W. B. FITE ; Theory of functions of a real variable, two hours. — By Dr. F. R. SHARPE : Introduction to mathematical physics, three hours. — By Dr. A. RANUM : Algebra, three hours (first term). — By Dr. D. C. GILLESPIE : Series, three hours (first term) ; Linear differential equations, three hours (second term). — By Dr. C. F. CRAIG : Analytic geometry, three hours. — By Dr. F. W. OWENS : Differential equations, two hours. — By Mr. J. V. MCKELVEY : Projective geometry, three hours. The Oliver mathematical club will meet weekly.

UNIVERSITY OF ILLINOIS. (All courses are three hours a week.) — By Professor S. W. SHATTUCK : Differential equations (first semester) ; calculus of variations (second semester). By Professor E. J. TOWNSEND : Theory of functions of a complete variable. — By Professor G. A. MILLER : Theory of groups (second course). — By Professor E. J. WILCZYNSKI : Projective differential geometry. — By Professor H. L. RIETZ : Theory of statistics. — By Professor C. N. HASKINS : Fourier series ; advanced calculus. — By Professor J. W. YOUNG : Theory of automorphic functions. — By Dr. C. H. SISAM : Theory of invariants and higher plane curves. — By Dr. A. R. CRATHORNE : Partial differential equations of physics. — By Professor YOUNG or Dr. R. L. BÖRGER : Projective geometry and linear transformations.

YALE UNIVERSITY. — By Professor J. PIERPONT : Theory of functions of a complex variable, two hours. — Advanced theory of functions, two hours ; Elliptic functions, two hours. — By Professor P. F. SMITH : Transformations of space, two hours ; Differential geometry, two hours. — By Professor E. W. BROWN : Mechanics, two hours ; Advanced calculus, three hours ; Hydromechanics, two hours. — By Professor H. E. HAWKES : Projective geometry, two hours ; Advanced algebra, two hours. — By Dr. W. A. GRANVILLE : Elementary differential geometry, two hours. — By Dr. W. R. LONGLEY : Differential equations, two hours. — By Dr. G. M. CONWELL : Elementary differential equations, one hour ; Foundations of geometry, two hours ; Invariants, two hours. — By Dr. E. G. BILL : Integral equations, one hour ; Analytic geometry, two hours.

AT the technical school of Zurich Drs. DU PASQUIER, KIENAST, and MEISSNER have been appointed docents in mathematics.

DR. O. SPIESS, of the University of Basel, has been promoted to an associate professorship of mathematics.

PROFESSOR G. LORIA, of the University of Genoa, has been elected foreign member of the royal Bohemian academy of sciences.

DR. ZORETTI has been appointed lecturer under the conditions of the Peccot foundation at the Collège de France. The subject of his course is singular points of analytic functions.

PROFESSOR G. HERGLOTZ, of the technical school at Vienna, has accepted a professorship of mathematics at the University of Leipzig, as successor to the late Professor W. Scheibner.

PROFESSOR F. SCHILLING, of the technical school at Danzig, has declined the professorship of mathematics at the technical school at Dresden made vacant by the resignation of Professor Disteli.

DR. A. THAER has been appointed docent in mathematics at the University of Jena.

PROFESSOR W. LUDWIG, of the technical school at Braunschweig, has accepted the professorship of descriptive geometry at the technical school at Dresden.

PROFESSOR V. VOLTERRA, of the University of Rome, has been elected corresponding member of the imperial academy of sciences of St. Petersburg.

PROFESSOR P. BURGATTI, of the University of Bologna, has been elected to membership in the Bologna academy of sciences.

DR. E. E. LEVI has been appointed associate professor of the calculus at the University of Genoa.

AT Columbia University, Dr. G. H. LING has been promoted to an associate professorship of mathematics.

AT Stanford University Professor H. C. MORENO and Professor S. D. TOWNLEY have been promoted to associate professorships of applied mathematics.

Mr. J. V. McKelvey has been appointed instructor in mathematics at Cornell University.

The alumni association of Columbia University gave a dinner to Dean J. H. Van Amringe, on April 3, to celebrate his birthday and a half century of teaching in Columbia College.

Professor J. J. Browne, of the Colorado School of Mines died February 20, at the age of 42 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

Bücher, neue, über Naturwissenschaften und Mathematik. Mitgeteilt Winter 1908–09. Leipzig, Hinrich. 8vo. Pp. 65–86. M. 0.30

Burkhardt (H.). See Enzyklopädie.

Castells Vidal (P.). Balanza algebraica para obtener las raíces reales de las ecuaciones algebraicas ó trascendentes, con una incógnita. Conferencia dada en el Instituto de Ingenieros Civiles el día 28 de Octubre de 1908. Madrid, 1909. 22 pp. P. 1.00

Correale (E.). Principali formole di geometria analitica in coordinate biangolari. Napoli, De Rubertis, 1908. 16mo. 23 pp.

Enzyklopädie der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen. Vol. II: Analysis. In 3 Teilen. Redigiert von H. Burkhardt und W. Wirtinger. 3ter Teil. 1tes Heft. Leipzig, Teubner, 1909. 8vo. Pp. 1–46. M. 1.50

Hoffmann (C.). Das Abelsche Theorem für die elliptischen Integrale. (Diss.) Tübingen, 1909. 8vo. 47 pp.

Junker (F.). Höhere Analysis. 1er Teil: Differentialrechnung. 3te, verbesserte Auflage. Neudruck. (Sammlung Göschen, 87). Leipzig, Göschen, 1908. 8vo. 204 pp. Cloth. M. 0.80

Koestler (W.). Beiträge zu Reihen-Entwicklungen nach Bessel'schen Zylinder-Funktionen. (Diss.) Bern, 1907. 8vo. 110 pp.

Laisant (C. A.). See Lefebvre (L. C.).

Lefebvre (L. C.). Notions fondamentales de la théorie des probabilités. Préface de C. A. Laisant. Paris, Dulac, 1909. 8vo. 3 + 181 pp.

Neumann (A.). Ueber quadratische Verwandtschaften in Ebene und Raum, insbesondere Kreis- und Kugelverwandtschaft. (Diss.) Königsberg, 1908. 8vo. 82 pp.

Pagliano (C.). Sulle varietà algebriche formate da una semplice infinità di spazi. Cuneo, 1908. 4to. 10 pp.

Pezzo (P.). Lezioni di geometria proiettiva, dettate nell' università di Napoli nell' anno 1908–09. Napoli, De Rubertis, 1908. 8vo. 354 pp. L. 12.00

Schmidt (R.). Ueber zweite Polarflächen einer allgemeinen Fläche 4ter Ordnung. (Diss.) Breslau, 1908. 8vo. 96 pp.

Wirtinger (W.). See Enzyklopädie.

II. ELEMENTARY MATHEMATICS.

Bos (H.) et Rebière (A.). Eléments de géométrie. 6e édition. Paris, Hachette, 1909. 8vo. 503 pp. Fr. 7.50

Clement (L.). The ancient science of numbers. London, Fowler, 1909. 12mo. Cloth. 3s. 6d.

Elementos de aritmética con algunas nociones de algebra por los hermanos de las escuelas cristianas. 10a edición. Correspondiente á los cursos medio y superior. Paris, Procuraduria general, 1909. 16mo. 392 pp.

Games (H. G. W. H.). See Nettell (R.).

Jiménez Rueda (C.). Lecciones de geometria métrica en que se contesta á los programas de las escuelas y de la Facultad de Ciencias. 2a edición aumentada. Madrid, Suárez, 1909. 778 pp. P. 25.00

Knight (W. A.). Cassell's elementary geometry. London, Cassell, 1909. 8vo. 262 pp. Cloth. 2s. 6d.

Matriculation model answers: Mathematics. Being the London University matriculation papers in mathematics from June, 1906 to January, 1909. London, Clive, 1909. 8vo. 144 pp. 2s.

Nettell (R.) and Games (H. G. W. H.). Exercises in algebra. With answers. London, Longmans, 1909. 8vo. 318 pp. Cloth. 5s. 6d.

Nouvelles tables de logarithmes à cinq décimales pour les lignes trigonométriques dans les deux systèmes de la division centésimale et de la division sexagésimale du quadrant et pour les nombres de 1 à 12,000. Edition spéciale à l'usage des candidats aux Ecoles polytechnique et de Saint-Cyr. Paris, Gauthier-Villars, 1908. 8vo. Fr. 3.00

Rebière (A.). See Bos (H.).

Schwering (K.). 100 Aufgaben aus der niederen Geometrie nebst vollständigen Lösungen. 3te, verbesserte Auflage. Freiburg, Herder, 1908, 8vo. 12 + 170 pp. M. 2.60

Smith (D. E.). See Wentworth (G.).

Socci (A.) e Tolomei (G.). Aritmetica generale e algebra: libro di testo per i licei. Vol. I, per la 1a classe. Firenze, Le Monnier, 1909. 8vo. 8 + 150 pp. L. 1.50

Stassano (P.). Elementi di geometria. Napoli, Pellerano, 1909. 16mo. 12 + 208 pp. L. 2.50

Tolomei (G.). See Socci (A.).

Unwin (P. W.). Practical solid geometry. London, Bell, 1909. 8vo. 280 pp. Cloth. 4s. 6d.

Wolff (H.). Sätze und Aufgaben der Geometrie für Realanstalten, nach den preussischen Lehrplänen von 1901 zusammengestellt. 2ter Teil. Pensum der Oberstufe (Obersekunda und Prima). Leipzig, Teubner, 1909. 8vo. 4 + 120 pp. M. 2.00

Weill (A.). Sammlung graphischer Aufgaben für den Gebrauch an höheren Schulen. I. Mathematik. Gebweiler, Boltze, 1909. 8vo. 64 pp. M. 1 80

Wentworth (G.) and Smith (D. E.). Complete arithmetic. Boston, Ginn, 1909. 8vo. 6 + 474 pp. Cloth. $0.70

III. APPLIED MATHEMATICS.

DANELLE (M. R.). Géométrie cotée (classe de troisième B). Paris, Colin, 1908. 16mo. 64 pp. Fr. 1.25

DURAND (W. F.). The resistance and propulsion of ships. 2nd edition, revised. New York, Wiley, 1909. 8vo. 13 + 427 pp. Cloth. $5.00

FERVAL (H.). Eléments de géométrie descriptive à l'usage des candidats aux baccalauréats de l'enseignement secondaire et aux écoles du gouvernement. 9e édition, complètement refondue, conformément aux programmes officiels du 27 juillet 1905. Paris, Belin, 1908. 12mo. 350 pp. Fr. 3.75

MILLAR (A. V.). See PHILLIPS (J. D.).

PHILLIPS (J. D.) and MILLAR (A. V.). Essentials of descriptive geometry. Madison, Wis., Tracy, 1909. 12mo. 7 + 104 pp. Cloth. $1.00

PRELINI (C.). Graphical determination of earth slopes, retaining walls, and dams. London, Constable, 1909. 8vo. 8s.

REIFF (W.). Rollen einer Kugel in einem Zylinder ohne Einwirkung der Schwerkraft. (Diss.) Tübingen, 1908. 8vo. 36 pp.

ROSENTHAL (L. W.). Practical calculation of transmission lines, for distribution of direct and alternating currents by means of overhead, underground, and interior wires for purposes of light, power, and traction. New York, McGraw, 1909. 8vo. 10 + 93 pp. Cloth. $2.00

SALMOIRAGHI (A.). Nouvelles tables des coordonnées rectangulaires à cinq et à quatre décimales calculées suivant la division sexagésimale du quadrant mais à subdivision décimale du degré, et tables auxiliaires pour les calculs de la tachéometrie. Milano, Peggiani, 1909. 4to. 15 + 229 pp.

WEISSE (E.). Anwendung der elliptischen Funktionen auf ein Problem aus der Theorie der Gelenkmechanismen. (Diss.) Rostock, 1907. 8vo. 102 pp.

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THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and forty-third regular meeting of the Society was held in New York City on Saturday, April 24, 1909, extending through a morning and an afternoon session. About thirty-five persons were in attendance, including the following twenty-six members of the Society:

Professor Maxime Bôcher, Professor A. S. Chessin, Miss E. M. Coddington, Professor F. N. Cole, Dr. E. B. Cowley, Dr. F. J. Dohmen, Professor L. P. Eisenhart, Professor W. H. Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Mr. W. C. Krathwohl, Professor G. H. Ling, Dr. E. N. Martin, Professor G. D. Olds, Professor W. F. Osgood, Mr. H. W. Reddick, Professor D. E. Smith, Professor P. F. Smith, Dr. W. M. Strong, Mr. J. S. Thompson, Professor H. W. Tyler, Professor Oswald Veblen, Mr. H. E. Webb, Miss M. E. Wells, Professor H. S. White, Professor J. E. Wright.

The President of the Society, Professor Maxime Bôcher, occupied the chair, being relieved at the afternoon session by Vice-President Professor Edward Kasner. The Council announced the election of the following persons to membership in the Society: Miss W. B. Bauer, High School, Topeka, Kan.; Mr. W. W. Denton, University of Illinois; Mr. Meyer Gaba, University of Kansas; Professor W. H. Garrett, Baker University; Miss S. E. Graham, High School, Topeka, Kan.; Mr. G. F. Gundelfinger, Sheffield Scientific School; Professor W. A. Harshbarger, Washburn College; Mr. L. T. W. Hogrefe, Milwaukee, Wis.; Professor L. A. Howland, Wesleyan University; Mr. George Melcher, State Normal School, Springfield, Mo. Three applications for membership in the Society were received.

Professor H. S. White was reelected a member of the Editorial Committee of the Transactions for a term of three years beginning October 1, 1909.

The following papers were read at this meeting:

(1) Professor R. D. CARMICHAEL: "On certain functional equations."

(2) Professor R. D. CARMICHAEL: "Note on some polynomials related to Legendre's coefficients."

(3) Professor W. H. Jackson: "A theorem concerning simple continued fractions."

(4) Professor W. H. Jackson: "Shadow rails."

(5) Professor L. P. Eisenhart: "The twelve surfaces of Darboux and the transformation of Moutard."

(6) Professor W. F. Osgood: "On Cantor's theorem concerning the coefficients of a convergent trigonometric series, with generalizations."

(7) Mr. L. S. Dederick: "Certain singularities of transformations of two real variables."

(8) Professor Paul Saurel: "On Fredholm's equation."

(9) Professor J. E. Wright: "On abelian functions of genus 3."

(10) Dr. J. C. Morehead: "A simplification of Lagrange's method for the solution of numerical equations (second paper)."

(11) Professor A. S. Chessin: "On gyroscopic couples."

(12) Professor Edward Kasner: "The interpretation of differential equations in line coordinates."

(13) Professor E. D. Roe, Jr.: "On the extension of the exponential theorem."

Mr. Dederick's paper was presented to the Society through Professor Bouton. In the absence of the authors Mr. Dederick's paper was read by Professor Bôcher, and the papers of Professors Carmichael, Saurel, Chessin, Kasner, and Roe, and Dr. Morehead were read by title. Professor Saurel's paper appears in full in the present number of the Bulletin. Abstracts of the other papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper Professor Carmichael considers the equations

$$h(x+y)\,h(x-y) = [h(x)]^2 + [h(y)]^2 - c^2,$$

$$g(x+y)\,g(x-y) = [g(x)]^2 - [g(y)]^2.$$

He finds that the general solution of the first is

$$h(x) = \pm \tfrac{1}{2}c(e^{kx} + e^{-kx})$$

and shows that, except for a constant factor, several related functional equations have the same solution. The second of the given equations has two solutions

$$g(x) = \pm \frac{g'(0)}{2k}(e^{kx} - e^{-kx}), \quad \text{if} \quad k \neq 0,$$

$$g(x) = \pm g'(0)\ x, \quad \text{if} \quad k = 0,$$

where $g'(0)$ is an arbitrary constant.

2. In his second paper Professor Carmichael considers a class of functions

$$P_{na} = \frac{1}{2^n n\,!}\,\frac{d^n}{dx^n}(x^{2a} - 1)^n,$$

and shows that several properties of Legendre's polynomials are properties also of this generalized function. P_{n1} is the usual Legendre coefficient.

3. In his first paper Professor Jackson considers all those integral multiples of a given number for which the fractional part lies between given limits. It is shown that the integral multipliers satisfying this condition possess a periodic formation, one period within another, but that these periods are subject to breaks at calculable intervals.

4. In his second paper Professor Jackson applies the results of his first to calculate the position of the shadow rails observed when one set of rails is viewed through another. It is established that the spacing of the shadow bands depends only on the spacing of the two sets of rails and not on their breadths, and that when the position of the observer is altered the shadow bands move with magnified velocity in the same direction as that of the apparent motion of the more closely spaced set of rails relative to the other set.

5. The problem of the infinitesimal deformation of a surface S is equivalent to the determination of a surface S_1 corresponding to S with orthogonality of linear elements, or of an associate surface S_0 whose tangent plane is parallel to the corresponding tangent plane to S and whose asymptotic lines correspond to a conjugate system on S. In the fourth volume of his Leçons Darboux shows that nine other surfaces are associated with S, S_1, S_0 in such a way that any three successive surfaces of the closed cycle are in relations similar to those which hold between S, S_1, and S_0. Darboux is led to the result that the cycle is

closed after he has calculated the coordinates of the various surfaces. Professor Eisenhart, in the first part of his paper, is concerned with the reasons for this closure and with an investigation as to the possibility that all of the twelve surfaces be not distinct. In the second part of his paper, he considers the group of surfaces which figure in the infinitesimal deformations of a surface S and the three surfaces S', S'', S''' which arise from S by transformations of Moutard,* these surfaces being such that the pairs S, S'; S, S''; S', S'''; S'', S''' are the focal surfaces of W-congruences. In general there are sixteen distinct surfaces in the group. An exceptional case of importance is that for which the two surfaces associate to S are associate to one another.

6. Professor Osgood's paper contains a simple proof of the theorem that if a trigonometric series

$$\Sigma(a_n \cos nx + b_n \sin nx)$$

converges for all values of x in a certain interval, however short, its coefficients approach zero as their limit. The theorem is generalized to series of the form $\Sigma A_n f_n(x)$, where f is continuous in an interval (a, b) and in every subinterval (α, β) takes on values numerically greater than a certain positive constant l at some point of the interval, provided $n \geqq m$. If such a series converges in (a, b), then

$$\lim_{n=\infty} A_n = 0.$$

Further generalizations to multiple series

$$\Sigma f_{m,n}(x, y), \quad \Sigma f_{m,n,p}(x, y, z),$$

are given.

The paper will be offered for publication in the *Transactions*.

7. The object of Mr. Dederick's paper is to discuss general properties of those singularities of transformations of two real variables $u = \phi(x, y)$, $v = \psi(x, y)$ which are due not to discontinuities in the functions ϕ or ψ, or their partial derivatives, but to the vanishing of the Jacobian determinant $J = \phi_x\psi_y - \phi_y\psi_x$. The most important results are the derivation and use of a general formula for the transformation of the derivative of one variable with respect to the other, which applies with an unimportant exception to all singularities of

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* Bianchi, Lezioni, vol. 2, p. 69.

this sort, and the classification of such singularities on the basis of this formula. The transformed second derivative is in general infinite except in the simplest case. Some geometric properties discussed are those of the critical slopes, the preservation of regularity of an arc, the criterion for the transformation of a curve into a point, and the order of contact of the curves $\phi = \text{const.}$ and $\psi = \text{const.}$ with each other and with $J = 0$. A reduction of the transformation to the normal form, $u_1 = x_1$, $v_1 = y_1^2$, is obtained for any neighborhood of an arc of $J = 0$ in which one of the expressions $\phi_x J_y - \phi_y J_x$ and $\psi_x J_y - \psi_y J_x$ remains different from zero, the reduction being effected by transformations of the two planes whose Jacobians nowhere vanish. The following theorem on implicit functions is derived: If $F(u, x)$ and all its partial derivatives of order less than n vanish at (u_0, x_0) and the partial derivatives of the nth order are continuous near (u_0, x_0) and not all zero at that point, then to every real simple root of the equation of the nth degree for formally determining du/dx, corresponds a solution $u = f(x)$ which has a continuous derivative near (u_0, x_0).

9. Professor Wright's paper is concerned with the determination of the differential equations satisfied by abelian θ functions of genus 3. The equations are obtained in covariantive form, and they lead to the generalization of Kummer's quartic surface. This turns out to be a three spread in space of six dimensions, of the 8th order in the hyperelliptic, of the 16th order in the non-hyperelliptic case.

10. In an earlier paper* Dr. Morehead showed how binomial synthetic division may be applied to the approximate expression of the incommensurable real roots of numerical equations in terms of simple continued fractions

$$a_1 + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n}.$$

This method is modified, in the present paper, so as to give the approximate expression of the roots in continued fractions of the form

$$a_1 + \frac{b_2}{a_2} + \frac{b_3}{a_3} + \cdots + \frac{b_n}{a_n},$$

the b's being arbitrarily chosen.

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* Read before the Chicago Section, April 9, 1909.

The similar process for the expression of the roots in ascending continued fractions

$$a_1 + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots + \frac{a_n}{b_n}$$

is a generalization of Horner's method, as the above reduces to the decimal fraction $a_1\, a_2\, a_3 \cdots a_n$ for $b_2 = b_3 = \cdots = b_n = 10$.

11. The well known property of rapidly revolving bodies to resist efforts tending to deflect their principal axes is due to the rise of couples among which that of the so-called turning forces produces most of the effect. This couple is generally assumed to be equal to $C\omega\phi' \sin\theta$ where C, ω, ϕ, and θ denote respectively the moment of inertia about the axis of spin, the angular velocity of deflection, the angular velocity of spin S, the angle of inclination between the axis of spin and the axis of deflection. In the present paper Dr. Chessin shows that the correct expression of the turning couple is given by the following values of its projections for the three eulerian axes:

$$C\omega(\phi' + \omega \cos\theta) \sin\theta, \quad (2A - C)\omega\theta' \cos\theta, \quad C\omega\theta' \sin\theta,$$

which reduce to the commonly used approximate value $C\omega\phi' \sin\theta$ when $\theta' = 0$ and ω is very small compared to ϕ'.

12. An oriented straight line may be defined most simply by means of its Hessian coordinates (u, v), the latter representing the length of perpendicular dropped from the pole and the former representing the angle which this perpendicular makes with the initial line. A differential equation of the form $F(dv/du, u, v) = 0$ then defines a single infinity of oriented curves. The interpretation of derivatives is very simple. Professor Kasner discusses in particular the systems of curves represented by linear and Riccati equations. These types may be regarded as natural extensions of systems of parallel curves. In constrast with the usual statements for cartesian coordinates (x, y), the results are here invariant under the displacement group.

13. Professor Roe's paper contains an elaboration of the theory of the exponential function.

F. N. Cole,
Secretary.

APRIL MEETING OF THE CHICAGO SECTION.

THE twenty-fifth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Chicago, on Friday and Saturday, April 9–10, 1909. The extensive programme required four full half-day sessions. Professor G. A. Miller, Chairman of the Section presided except on Friday afternoon when Professor E. B. Van Vleck, Vice-President of the Society, occupied the chair. The attendance at the various sessions included sixty-two persons among whom were the following forty-six members of the Society:

Mr. R. P. Baker, Mr. W. H. Bates, Dr. G. D. Birkhoff, Professor G. A. Bliss, Mr. Thomas Buck, Mr. H. E. Buchanan, Professor D. F. Campbell, Professor H. E. Cobb, Professor D. R. Curtiss, Professor L. E. Dickson, Dr. Arnold Dresden, Professor J. C. Fields, Professor W. B. Ford, Professor C. N. Haskins, Mr. T. H. Hildebrandt, Professor T. F. Holgate, Professor Kurt Laves, Dr. A. C. Lunn, Dr. E. B. Lytle, Professor Max Mason, Professor Malcolm McNeill, Mr. E. J. Miles, Dr. W. D. MacMillan, Professor G. A. Miller, Professor E. H. Moore, Dr. R. L. Moore, Dr. J. C. Morehead, Professor F. R. Moulton, Dr. L. I. Neikirk, Mrs. Anna J. Pell, Professor Alexander Pell, Professor C. A. Proctor, Professor H. L. Rietz, Mr. A. R. Schweitzer, Professor C. S. Slichter, Professor G. T. Sellew, Professor J. B. Shaw, Professor H. E. Slaught, Professor E. J. Townsend, Dr. A. L. Underhill, Professor E. B. Van Vleck, Professor E. J. Wilczynski, Professor A. E. Young, Professor J. W. Young, Professor J. W. A. Young, Professor Alexander Ziwet.

On Friday evening forty members of the Society dined together in the café of the university commons and discussed informally various topics of interest, including the plans for the summer, 1909, meeting of the British association for the advancement of science to be held at Winnipeg, and the next International congress of mathematicians to be held in England in 1912. Professor J. C. Fields, of Toronto, extended a cordial invitation to the members to attend the former, and Professor E. H. Moore urged upon all the desirability of laying plans now to attend the latter.

Particular interest centered in the report of the committee appointed at the January, 1909, meeting to consider plans for improvement in the methods of making mathematical appointments in this country. In case a suitable plan had approved itself to the committee and to the Section, it was the intention to present this plan to the Council for its consideration. The committee, after careful deliberation, reached the conclusion that, while such improvement is manifestly desirable in many American colleges and universities, they were unable to recommend for consideration any plan for such improvement by action of the Society. The Section accepted the report and requested the Secretary to arrange for the publication of a plan which was proposed tentatively by Professor Wilczynski, chairman of the committee, and which he described in some detail. This will appear later in the BULLETIN.

The following papers were read at this meeting:

(1) Professor C. N. HASKINS: "On a class of discontinuous functions of two variables."

(2) Mr. A. E. WESTERN and Dr. J. C. MOREHEAD: "The Fermat number $2^{256} + 1$."

(3) Dr. J. C. MOREHEAD: "A simplification of Lagrange's method for the solution of numerical equations."

(4) Mr. THOMAS BUCK: "Oscillations near Lagrange's equilateral triangle points in the problem of three bodies."

(5) Professor L. E. DICKSON: "On the equivalence of pairs of quadratic forms under rational transformation."

(6) Dr. ARTHUR RANUM: "The group of classes of quadratic integers with respect to a composite ideal as modulus."

(7) Professor A. E. YOUNG: "On surfaces having isotherm-conjugate lines of curvature."

(8) Professor W. B. FORD: "A set of criteria for the summability of divergent series."

(9) Professor W. B. FORD: "On the determination of the asymptotic developments of a given function."

(10) Mr. H. E. BUCHANAN: "A class of periodic orbits of three finite bodies."

(11) Professor G. A. MILLER: "Automorphisms of order two."

(12) Mr. R. P. BAKER: "The attack on the space and time concepts by Einstein and Minkowski."

(13) Professor E. W. DAVIS: "An imaginary conic."

(14) Dr. L. I. NEIKIRK: "Groups of rational integral transformations in a general field."

(15) Professor L. E. DICKSON: "A theory of invariants."

(16) Professor L. E. DICKSON: "Combinants."

(17) Mr. W. H. BATES: "Maschke's symbolic method applied to some questions in geometry of hyperspace."

(18) Professor L. D. AMES: "A simpler proof of Lie's theorem for ordinary differential equations."

(19) Dr. W. D. MACMILLAN: "Periodic orbits about an oblate spheroid."

(20) Professor H. L. RIETZ: "On the effect of types of correspondence on Bravais's coefficient of correlation."

(21) Professor F. R. MOULTON: "Oscillating satellites when the finite bodies describe elliptic orbits."

(22) Professor E. J. WILCZYNSKI: "Projective differential geometry of developables."

(23) Mr. ARTHUR PITCHER: "Complete elementary theory of certain properties of classes of functions."

(24) Mr. T. H. HILDEBRANDT: "Remarks on the general theory of point sets."

(25) Mrs. ANNA J. PELL: "Biorthogonal systems."

(26) Professor F. R. MOULTON and Dr. W. D. MACMILLAN: "On the solution of linear differential equations with periodic orbits."

Mr. Pitcher was introduced by Professor Moore. In the absence of the authors the papers of Professors Ames and Davis and Dr. Ranum were read by title. Abstracts of the papers follow below. The numbering corresponds to the titles in the above list.

1. The note of Professor Haskins calls attention to functions of the types

$$Z(x, y) = \frac{yu(x)}{y^2 + [u(x)]^2}, \quad \text{and} \quad Z(x, y) = \frac{ye^{-\frac{1}{[u(x)]^2}}}{y^2 + e^{-\frac{2}{[u(x)]^2}}},$$

where $\mathop{\mathrm{L}}\limits_{x=0} u(x) = 0$. The first of these functions has the property that

$$\mathop{\mathrm{L}}\limits_{x=0} Z(x, mv(x)) = 0 \quad \text{if} \quad \mathop{\mathrm{L}}\limits_{x=0} \frac{u(x)}{v(x)} = 0 \quad \text{or} \quad \mathop{\mathrm{L}}\limits_{x=0} \frac{v(x)}{u(x)} = 0,$$

and

$$\mathop{\mathrm{L}}_{x=0} Z(x, mv(x)) = \frac{m\alpha}{m^2+\alpha^2} \quad \text{if} \quad \mathop{\mathrm{L}}_{x=0} \frac{u(x)}{v(x)} = \alpha \neq 0.$$

This gives rise to a set of functions which show the discontinuity in question along systems of curves having osculation with $y = 0$ of continually increasing order at the origin.

2. This paper describes the method of computation followed by Mr. Western and Dr. Morehead which resulted in showing that the Fermat number $F_8 = 2^{256} + 1$ fails to satisfy the criterion for Fermat primes,*

$$3^{2(2^n-n-2)} \equiv \pm 2^x(2^{2^{n-1}} \pm 1) \mod F_n, \quad (F_n = 2^{2^n} + 1),$$

and is therefore composite. The result is especially interesting as completing a chain of five composite Fermat numbers, F_5, F_6, F_7, F_8, F_9, following the first five, and only known Fermat primes, 3, 5, 17, 257, 65537.

3. In this paper Dr. Morehead explains the application of binomial synthetic division to the approximate expression of an incommensurable real root of a numerical equation,

$$f_1(x) = c_0x^m + c_1x^{m-1} + \cdots + c_m = 0,$$

in terms of a continued fraction of the form

$$(a_1, a_2, a_3, \cdots a_n) = a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{+a_n}.$$

The quotients $a_1, a_2, a_3, \cdots, a_n$ are the integral parts of corresponding roots of the equations

$$\begin{aligned} f_1(x_1) &= c_1x_1^m + c_1x_1^{m-1} + \cdots + c_m = 0, \\ f_2(x_2) &= c_0''x_2^m + c_1''x_2^{m-1} + \cdots + c_m'' = 0, \\ f_3(x_3) &= c_0'''x_3^m + c_1'''x_3^{m-1} + \cdots + c_m''' = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ f_n(x_n) &= c_0^{(n)}x_n^m + c_1^{(n)}x_n^{m-1} + \cdots + c_m^{(n)} = 0, \end{aligned}$$

respectively, where each equation $f_{k+1}(x_{k+1}) = 0$ is connected with the preceding $f_k(x_k) = 0$ by the relation $x_k = a_k + 1/x_{k+1}$, or $x_{k+1} = 1/(x_k - a_k)$. From the latter relation it is easily seen that the coefficients of $f_{k+1}(x_{k+1})$ may be obtained from those of $f_k(x_k)$ if we diminish the roots of $f_k(x_k)$ by a_k, using the binomial synthetic division process as in Horner's method, and reverse the order of the coefficients so obtained.

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* A modification, due to A. E. Western, of the Pepin criterion for the base 3 : $3^{2^{(2^n-1)}} \equiv -1 \mod F_n$.

Approximations for the roots of numerical equations may be obtained by this means more easily and more rapidly, in general, than by Horner's method. Thus for example, without using more than six figures in any of the coefficients, we obtain for the root lying between 1 and 2 of the equation,

$$2x^3 - x^2 + 2x - 4 = 0,$$

the approximation (1, 6, 1, 3, 1, 1, 3, 2, 3, 1, 4, 1, 1, 3, 1, 1, 5, 2, 1, 2, 2, 3) = 82994039/72327612, which is correct to 2 in the sixteenth decimal place.

4. In the first part of this paper Mr. Buck considers certain periodic orbits of an infinitesimal body attracted by two finite bodies revolving in circles about their center of mass. Convenient parameters are introduced and by a discussion of the linear terms the generating solutions are found. All the terms of the differential equations are then considered and the question of the existence of the continuations of the generating solutions is discussed.

In the second part of the paper certain periodic orbits of three finite bodies near Lagrange's equilateral triangle points are considered. Parameters are introduced as before and the linear terms discussed. The discussion when the higher terms of the differential equations are considered has not yet been completed.

5. The first paper by Professor Dickson considers the equivalence of two pairs of quadratic forms ϕ, ψ and ϕ', ψ' under linear transformation in a field F. As in the algebraic theory, the elementary divisors of $\phi - \lambda\psi$ must be the same as those of $\phi' - \lambda\psi'$. But this condition is not in general sufficient for equivalence in F. We extend the field F to a field F' by adjoining the roots λ_i of $|\phi - \lambda\psi| = 0$. If ϕ_1, ψ_1 is a canonical type of ϕ, ψ within F', each transformation effecting the reduction is the product of one with conjugate variables by an automorph of ϕ_1, ψ_1. A discussion of the transformations of ϕ_1, ψ_1 into ϕ_1', ψ_1' leads to conditions for their equivalence within F'. The conjugacy of the variables enables us to decide upon the equivalence of the initial pairs within F. Use

is made of the work of Kronecker and a modified form of that of Weierstrass. Both in the singular and in the non-singular cases, simple necessary and sufficient conditions are obtained for the equivalence of two pairs of forms under linear transformation in the given field. This paper has been offered for publication in the *Transactions*.

6. In the ordinary theory of rational numbers the group of classes of congruent integers with respect to a composite modulus is well known and is called by Weber is his Algebra the most important example of a finite abelian group. In the more general theory of numbers in an algebraic field the corresponding group of classes of integers with respect to a composite ideal does not seem to have been studied heretofore. In this paper a beginning is made by Dr. Ranum. For the special case of a quadratic field $(\sqrt{m})$ he determines the structure of the group, and finds a system of independent generators. If the modulus is a power of a prime ideal p, the latter being a factor of the principal ideal (p), the group is one of nine distinct types, depending on the value of m, the grade of p, and the value of the rational prime p $(p = 2,\ p = 3, \text{ or } p > 3)$. If the modulus contains as factors two or more distinct prime ideals, the group is a direct product of two or more groups of these nine types. The cases in which the group is cyclic, and therefore primitive roots exist, are enumerated.

7. In a paper published in the January issue of the *Transactions*, Professor Young discussed a class of surfaces characterized by the two properties that their lines of curvature form a network of infinitesimal squares, and their asymptotic lines form a network of rhombuses. In the present paper he discusses the more general class, characterized by the latter of these two properties. The general problem of determining these surfaces is reduced to the simultaneous solution of two partial differential equations of the second order which involve two functions. Particular solutions are found and the corresponding surfaces determined. It is shown that these surfaces compose the class which have isotherm-conjugate lines of curvature, using this term in the sense defined by Bianchi.

8. Professor Ford's first paper appears in full in the present number of the BULLETIN.

9. The object of Professor Ford's second paper is to prove, and show the range of applicability of, a general formula employed extensively of late years by Barnes and made basal by him in a series of memoirs upon the determination of the asymptotic developments of a given function in the neighborhood of the point ∞. The formula in question is *

$$\sum_{n=1}^{m-1}\phi(n) \sim Z + \int \phi(m)dm - \tfrac{1}{2}\phi(m) + \sum_{n=0}^{n=\infty}\frac{(-1)^n B_{n+1}}{(2n+2)!}\frac{d^{2n+1}\phi(m)}{dm^{2n+1}},$$

where Z is a certain constant and B_n the nth Bernoulli number. It is shown that the class of functions $\phi(n)$ to which the formula applies is well defined in that any such function must be developable in a prescribed way about the point $n = \infty$.

10. At the January meeting of the Section, Mr. Buchanan read a preliminary report on a certain class of periodic orbits of three finite bodies. He showed how to expand the right members of the differential equations and discussed the linear terms. In the present paper it is proposed to introduce certain convenient parameters, prove the existence of five different types of orbits, show how to construct these orbits, and state certain theorems concerning their geometrical properties.

11. The necessary and sufficient condition that an automorphism is of order 2 is that every commutator arising from the automorphism corresponds to its inverse under the automorphism. Hence it results that in every automorphism of order 2 some operator besides identity must correspond to its inverse. The independent generators of an abelian group of odd order may be so chosen that in any given automorphism of order 2 each of these generators corresponds either to itself or to its inverse. If all the commutators (besides identity) of a group are of order 2, every complete set of conjugate operators of order 2 generates an abelian group of type $(1, 1, 1, \cdots)$, and such a group involves an abelian subgroup which is the direct product of Sylow subgroups of all odd orders. Every group in which the orders of all the commutators are powers of the same prime number is solvable, and such a group involves only one Sylow subgroup whose order is a power of this prime;

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* *Philosophical Transactions*, vol. 199 A (1902), p 444.

12. The validity of the time and space concepts has been challenged by Einstein * and Minkowski.† Mr. Baker's paper is a critical examination of their theory.

Einstein's definition of time is insufficient; *fluit* must be added. Velocities in (ξ, η, ζ, τ) are non-associative in Einstein's system. Minkowski avoids this by his definition of "Eigenzeit."

The usual treatment of relativity depends on a transformation and its inverse, and on the ordinary dualism. T moves the object and T^{-1} the observer. If the dualism is not abandoned and O represents the ordinary transformation for uniform motion, the selection of OT^{-1} is only possible on a strict definition of *object* as a totality C_3 of light sources which act on moving according to Einstein's hypothesis, that is, are each a totality of pulses (C_1), the pulse being invariant under T and Einstein's source remaining a C_1 of pulses under O. Cross selection in the multiple infinities must be barred and each time t must be associated with its place.

With these restrictions the system becomes a single sense philosophy, the "sense organ" accomplishing the accurate timing of light signals by "synchronized clocks."

The selection of TT^{-1} as the pair is without special significance for relativity, though T of course has its very important significance for the problem of relative motion.

Since O does not preserve the equations of electrodynamics it is probable that Einstein's success with OT^{-1} is due to abstraction.

At any rate "absolut ruhender Raum" as applied to (x, y, z, t) is erroneous. When the transformation is obtained it turns out that there is no analytical distinction between this system and (ξ, η, ζ, τ) beyond the sign of v. Minkowski postulates a world. He asks for (1) continuity in four dimensions (x, y, z, t), (2) each point to be a substantial point, with assigned mass, (3) $ds/dt < V$, (4) a "Geschenk von oben," the contraction of moving bodies in the direction of their motion according to Einstein's law.

In the actual work "each" in (2) is defined by a continuous function, thus avoiding an obvious difficulty with the point set theory.

The contraction in (4) is not an affine transformation, the factor for the inverse transformation being the same as for the direct, and not the reciprocal of it. An ordinary hydrodynamic universe in (x, y, z, t) with an affine contraction for (4) has for its only rotating molecules those with fixed axis, size, and angular velocity, posited as eternal.

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* *Annalen der Physik*, vol. 17 (1905).

† *Physikalische Zeitschrift*, Feb., 1909. *Gottinger Nachrichten*, Dec., 1908.

The erroneous interpretation of the contraction is however a matter of indifference; satisfaction of the equations is all that is necessary.

13. In this paper Professor Davis points out that the conic $\tilde{x}^2 + \tilde{y}^2 e^{2i\phi} = 1$ can be gotten by revolving the intersection of the cylinder $x^2 + y^2 = 1$ with $z = iy$ about the x-axis through an angle ϕ, and then projecting the intersection upon the xy-plane. The distribution of the imaginary elements is indicated by a diagram.

14. In this paper Dr. Neikirk considers the totality of transformations of the type

$$T_{m,i} \equiv [x : \phi_{m,i}(x)],$$

where

$$\phi_{m,i}(x) = \sum_{j=0}^{j=m} \alpha_{j,i} x^{m-j} \qquad (\alpha_{0,i} \neq 0),$$

and the coefficients range independently over the elements or marks of a general field F.

The successive application of two of these transformations is a definite third one in the set. Symbolically $T_{m,i} T_{m',i'} = T_{mm',i''}$. They possess the group property. They also obey the associative law, if x is unrestricted. If x is restricted in range to a certain set, then the result of the action of $T_{m,i}$ on x, *i. e.*, on $\phi_{m,i}(x)$ must be in this same set for associativity.

Not all the transformations in this set have an inverse in the set. If $T_{m,i}$ $(m > 1)$ has $T_{m',i'}$ as an inverse, then x is restricted in range to the roots of an algebraic equation with coefficients in F, and $T_{m,i}$ gives a substitution on the roots of this equation. These transformations on the roots of an equation are a two-fold generalization of the substitution quantics given by Hermite, Dickson, and others. Several theorems are proved and several examples are given of groups generated by these transformations.

15. In the second paper by Professor Dickson, the invariants of a system of forms are investigated from the standpoint

of the classes of forms under the group of transformations of determinant unity. Whereas the discussion in the April number of the *Transactions* was limited to the case of a finite field, the present paper deals with an arbitrary field. For the case of the field of all real and complex numbers, the invariants considered are the rational integral invariants in the ordinary algebraic theory, as well as other single-valued invariants such as the rank of a quadratic form. Since all the single-valued invariants appear from a common standpoint, the present theory presents a synthesis of the special theories. The paper has been offered to the *American Journal of Mathematics.*

16. In the third paper by Professor Dickson, s-fold families $\Sigma x_i q_i$ of forms q_i on m variables ξ_j are separated into classes under the group Γ_1 of m-ary linear transformations on the ξ_j and the group G_1 of s-ary linear transformations on the x_i, each type of transformations having determinant unity. A function of the coefficients of the q_i is invariant under Γ_1, G_1 if and only if it takes the same value for all forms $\Sigma x_i q_i$ in any class. For a finite field the number of the resulting linearly independent invariants equals the number of classes. By an extension of a method given elsewhere,* it is determined which of these invariants are combinants; the number of linearly independent combinants is shown to equal the number of classes under Γ_1, G_1. For the practical determination of combinants various methods are developed. The general theory is applied to determine a fundamental system of five (ten) combinants of two binary (ternary) quadratic forms in the field of order p^n, $p > 2$. That the combinants actually form a fundamental system follows from a general theorem for finite and infinite fields (cf. preceding abstract). The paper has been offered to the *Quarterly Journal of Mathematics.*

17. In this paper, Mr. Bates uses Maschke's symbolic method to derive some of the formulas relating to angles between curves, isometric systems, and lines of curvature in hyperspace.

18. Professor Ames's paper appeared in full in the May BULLETIN.

19. The motion of a particle about an oblate spheroid is not, in general, periodic, but by virtue of the existence of one area

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* *Transactions*, April, 1909, § 1.

integral it is possible to reduce the problem from three dimensions to two. In the particular case of motion in the equatorial plane Dr. Macmillan shows that, within a certain range of variation of the parameters, the expression for the radius vector is always periodic. The expression for the rotation of the line of apsides has an interesting application to the motion of Jupiter's fifth satellite.

When the motion is not in the equatorial plane it is possible always to consider the motion in a revolving meridian plane. An orbit is considered to be periodic when its motion in this plane is periodic. It is found that the inclination can be taken arbitrarily and the initial velocity so determined that the orbit in the revolving plane is closed after the first revolution. These are the simplest types of orbits. Let us say that they have the period 2π. On varying the initial distance slightly it is found that a new period $2\pi/\lambda$ is introduced. If λ is rational, it is possible to determine the change in the initial velocity in terms of the initial displacement so that the orbit is reentrant after many revolutions. The integration of nonhomogeneous linear differential equations with periodic coefficients is involved in every step. These last solutions include five constants of integration out of the six necessary for a complete solution, with the condition that λ must be rational.

20. In this paper, Professor Rietz considers the effect of some different types of correspondence on results obtained by Bravais's formula for correlation. When a one to one correspondence exists for variates of the subject and relative classes, in treating the correlation of two attributes by statistical methods, the formula $r = \sum xy/n\sigma_x\sigma_y$ is uniquely defined. But a more general type of correspondence exists in some practical problems of statistics. A μ to ν correspondence occurs in which μ and ν are constant, but the μ or ν variates are not identical. In this case, it is a somewhat arbitrary matter as to which one of three or more distinct methods is used to obtain a one to one correspondence for application of the formula. It turns out that, with three different methods, which give significant measures of correlation, the values of r are in two of them different; and their numerical order is determined. In particular, if $\mu = 1$, although the correlation coefficients differ, the regression coefficients of the relative on the subject class are equal for two of the methods of correspondence; and this value

of the regression coefficient is the most probable value for the third method if the frequency distributions are symmetrical.

21. It is possible to project any three bodies so that they shall always be collinear and describe conic sections. Suppose one of the three is infinitesimal and that the orbits are ellipses. Then Professor Moulton shows that it is possible to displace the infinitesimal body from its position of equilibrium in an infinity of different ways, both in the plane of motion of the finite bodies and in three dimensions, so that its motion shall be periodic. When the finite bodies describe circles the orbit of the infinitesimal body, referred to rotating axes, re-enters after one revolution, but in the present case it re-enters only after many revolutions. The orbits depend upon a certain parameter and they are periodic for an infinite number of values of it, but these values do not constitute the continuous point set, for they must be such that certain commensurability conditions are satisfied.

22. In this paper Professor Wilczynski indicates a method for the development of a projective differential geometry of developable surfaces. These surfaces cannot be treated by the general methods of his previous papers, based upon the consideration of two linear homogeneous partial differential equations of the second order, because for a developable the corresponding system would become involutory, so that the most general integral would not be a projective transformation of any one. The difficulty is solved by adjoining an independent partial differential equation of the third order, to the involutory system of two equations of the second order. Mr. W. W. Denton is at present developing the details of this theory. It should be mentioned, however, that the dual theory, in which a developable is considered as an envelope of ∞^1 planes, is already contained in Professor Wilczynski's previous investigations.

23. The classes of functions considered by Mr. Pitcher are classes $\mathfrak{M}$ on a general class $\mathfrak{P}$ to $\mathfrak{A}$, where $\mathfrak{A}$ is the class of all real numbers. Certain ten properties of classes of functions have been found by Professor Moore to be very important in his work in general analysis. Four of these properties have to do with systems $(\mathfrak{A}, \mathfrak{P}, \mathfrak{M})$, while the others have to do with systems $(\mathfrak{A}, \mathfrak{P}, \Delta, \mathfrak{M})$ where Δ is a so-called development of $\mathfrak{P}$.

The objects of the present study are: (1) To investigate the elementary relations of dependence and independence of the ten properties mentioned above, (2) to give characterizations in terms of these properties of certain classes of elements $\mathfrak{P}$ and of functions $\mathfrak{M}$. So far the study has been confined to seven of the ten properties. These seven are not completely independent. Various characterizations for $\mathfrak{P}$ singular, $\mathfrak{P}$ dual, $\mathfrak{P}$ finite, and for certain classes of functions have been found.

24. A critical study of Fréchet's thesis * has revealed the fact that in order to obtain a theory of continuous functions, in particular the theory obtained by him, it is not necessary to condition the notion of limit further than to suppose it to be a relation between sequences of elements and single elements. In order to obtain a more general setting for the second part of his study, which concerns a distance function of pairs of elements, *voisinage*, a relation between pairs of elements and integers has been introduced. By the use of suitable conditions on this relation, Mr. Hildebrandt is able to obtain the entire theory of Fréchet and also the extension made by Hahn.† Other questions of independence are being investigated.

25. In Mrs. Pell's paper, necessary and sufficient conditions are found for the existence of the adjoint system of a given system of functions. The system is shown to be denumerable if the adjoint system exists. Conditions for the convergence of

$$\sum_{i=1}^{\infty}\int_a^b f(s)X_i(s)ds\int_a^b g(s)Y_i(s)ds$$

are deduced and it is shown that this sum is equal to

$$\int_a^b f(s)g(s)ds$$

when the given system is complete.

26. Special cases of the differential equations treated in this paper have arisen in the investigations of Professor Moulton and Dr. MacMillan on periodic orbits, and the methods of solving them were first developed in connection with these

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* "Sur quelques points du calcul fonctionnel," *Rend. del Cir. di Palermo*, vol. 22 (1906).

† Bemerkungen zu den Untersuchungen des Herrn Fréchet, *Monatshefte f. Math. u. Phys.*, vol. 19 (1908).

practical problems. This discussion is, in fact, reduced to that of periodic solutions of an associated set of differential equations. After giving a new derivation of the properties of the solutions in the general case, the class of special equations

$$(1) \qquad \frac{dx_i}{dt} = \sum_{j=1}^{n} \left[a_{ij} + \sum_{\kappa=1}^{\infty} \theta_{ij}^{(\kappa)}(t)\mu^{\kappa} \right] x_j \qquad (i = 1, \cdots, n)$$

is treated. The θ_{ij} are periodic in t with the period 2π. These equations include those which generally arise in practical problems.

When $\mu = 0$ let the roots of the characteristic equation of the resulting differential equations be $\alpha_1^{(0)}, \cdots, \alpha_n^{(0)}$. It is proved that if $\alpha_1^{(0)}, \cdots, \alpha_n^{(0)}$ are distinct and if no two of them differ by an imaginary integer, then the general solution of (1) is

$$(2) \qquad x_i = \sum_{j=1}^{n} A_j e^{\alpha_j t} y_{ij}(t) \qquad (i = 1, \cdots, n),$$

where the A_j are arbitrary constants, the y_{ij} are periodic with the period 2π, and

$$(3) \qquad \alpha_j = \alpha_j^{(0)} + \sum_{\kappa=1}^{\infty} \alpha_j^{(\kappa)} \mu^{\kappa}, \quad y_{ij} = \sum_{\kappa=0}^{\infty} y_{ij}^{(\kappa)} \mu^{\kappa}.$$

If two or more $\alpha_j^{(0)}$ differ by imaginary integers the solutions have *in general* the form (3), but there are interesting peculiarities both in the proof of their existence and in their construction.

If p of the $\alpha_j^{(0)}$ are equal, then in general the coefficients of $e^{\alpha_j t} y_{ij}(t)$ in (2) are polynomials in t and

$$(4) \qquad \alpha_j = \alpha_j^{(0)} + \sum_{\kappa=1}^{\infty} \alpha_j^{(\kappa)} \mu^{\kappa/p}, \quad y_{ij} = \sum_{\kappa=0}^{\infty} y_{ij}^{(\kappa)} \mu^{\kappa/p}.$$

In all the special cases which can arise convenient methods for constructing the solutions follow from the mode of treatment.

H. E. SLAUGHT,
Secretary of the Section.

A SET OF CRITERIA FOR THE SUMMABILITY OF DIVERGENT SERIES.

BY PROFESSOR WALTER B. FORD.

(Read before the Chicago Section of the American Mathematical Society, April 9, 1909.)

Introduction.

1. If the series

$$(1) \qquad u_0 + u_1 + u_2 + \cdots + u_n + \cdots$$

be divergent but of indeterminacy r in Cesàro's sense, its sum is defined by the relation

$$(2) \quad s = \lim_{n=\infty} \frac{f(n, r)s_0 + f(n-1, r)s_1 + \cdots f(1, r)s_{n-1} + f(0, r)s_n}{f(n, r) + f(n-1, r) + \cdots f(1, r) + f(0, r)},$$

where

$$f(n, r) = r(r+1)\cdots(r+n-1)/n! \ (r \geqq 1); \ s_n = u_0 + u_1 + \cdots + u_n$$

and where r is taken as the smallest integer for which the indicated limit exists. Since it is a necessary condition for indeterminacy r that

$$\lim_{n=\infty}\left(\frac{u_n}{n^r}\right) = 0\text{*}$$

it follows that but few series (1) are summable by (2) for any value of r however large, and hence that summability in Cesàro's sense is of relatively rare occurrence. It is therefore proposed in the present paper to show how an extended set of criteria, of which (2) affords a special type, may be constructed so as to provide an ascending scale of sharpness for the testing of summability. We shall show that such criteria may be constructed so as to bear a close analogy to the familiar logarithmic criteria for testing the convergence or divergence of a given series.

Formulation of Problem and Theorem.

2. In order to state the problem in more accurate terms we recall that the essential characteristic of the sum formula (2),

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*Cf. Bromwich, Infinite Series, p. 318 (1908).

or of any other general sum formula, is that it shall give $\lim_{n=\infty} s_n$ whenever the series (1) is convergent. Also we note that by increasing r in (2) we increase the order of infinitude of its denominator when $n=\infty$ and thereby increase the range of applicability of the formula. The following problem, which is the one we shall undertake, is therefore suggested: To determine an infinite sequence of functions

$$(3) \qquad f_0(n, r),\ f_1(n, r),\ f_2(n, r),\ \cdots,\ f_p(n, r),\ \cdots$$

each dependent upon the variable n and the parameter r and having the following characteristics:

$$(a) \qquad \lim_{n=\infty} \frac{\sum_{m=0}^{n} f_p(n-m, r) s_m}{\sum_{m=0}^{n} f_p(n-m, r)} = \lim_{n=\infty} s_n$$

whenever the series (1) is convergent;

$$(b) \qquad f_0(n, r) = \frac{r(r+1)\cdots(r+n-1)}{n!} \quad \text{(case of Cesàro);}$$

$$(c) \quad \lim_{n=\infty} \frac{\sum_{m=0}^{n} f_p(m, r)}{\sum_{m=0}^{n} f_p(m, r+1)} = 0, \qquad (d) \quad \lim_{n=\infty} \frac{\sum_{m=0}^{n} f_p(m, r)}{\sum_{m=0}^{n} f_{p+1}(m, 1)} = 0.$$

Any function of the sequence (3) will thus furnish an infinite subset of criteria having an increasing range of applicability with increasing r [see (c)], while in the case of a series (1) for which a given function of the sequence gives no meaning to the first member of (a) for any value whatever of r, we may at once obtain more powerful criteria by increasing p [see (d)]. Moreover, by virtue of (b) the extended set of criteria thus obtained will form a generalization of Cesàro's original formula (2).

In this connection, we shall now establish the following

THEOREM. *As a function $f_p(n, r)$ satisfying* (a), (b), (c), (d) *and therefore furnishing a set of criteria of summability for divergent series we may take*

$$f_p(n, r) = \frac{r(r+1)}{r+\log_p n} \prod_{n=2}^{n} \frac{r+\log_p n}{\log_p n},$$

where $\log_0 n = n$ *and* $\log_p n = \log\log\cdots\log n$ (*p times*); $p = 1, 2, 3, \cdots$.

3. *Proof.* We shall first establish the following lemma: Let $c_0, c_1, c_2, \cdots, c_n$ and $C_0, C_1, C_2, \cdots, C_n$ be two sequences of quantities of which the elements of the latter are real and positive. If $\lim\limits_{n=\infty} c_n = g$ and

$$(4) \qquad \lim_{n=\infty} \frac{C_n + C_{n-1} + \cdots + C_{n-m}}{C_n + C_{n-1} + \cdots + C_0} = 0, \ m = \textit{fixed integer} \geqq 0,$$

then

$$(5) \qquad \lim_{n=\infty} \frac{C_n c_0 + C_{n-1} c_1 + \cdots + C_0 c_n}{C_n + C_{n-1} + \cdots + C_0} = g.^*$$

In fact we shall then have $c_n = g + \epsilon_n$; $\lim\limits_{n=\infty} \epsilon_n = 0$, so that we may write

$$\frac{C_n c_0 + C_{n-1} c_1 + \cdots + C_0 c_n}{C_n + C_{n-1} + \cdots + C_0} - g = X(n) + Y(n),$$

where

$$X(n) = \frac{C_n c_0 + C_{n-1} c_1 + \cdots + C_{n-m} c_m - g(C_n + C_{n-1} + \cdots + C_{n-m})}{C_n + C_{n-1} + \cdots + C_0},$$

$$Y(n) = \frac{\epsilon_{m+1} C_{n-m-1} + \epsilon_{m+2} C_{n-m-2} + \cdots + \epsilon_n C_0}{C_n + C_{n-1} + \cdots + C_0}.$$

But if C represent the largest of the quantities $|c_0|, |c_1|, \cdots, |c_m|$, we shall have

$$|X(n)| \leqq (C + g) \frac{C_n + C_{n-1} + \cdots + C_{n-m}}{C_n + C_{n-1} + \cdots + C_0}$$

and hence $\lim\limits_{n=\infty} X(n) = 0$ by (4).

Again, corresponding to an arbitrarily small positive quantity ϵ we shall have

$$|Y(n)| \leqq \epsilon \frac{C_{n-m-1} + C_{n-m-2} + \cdots + C_0}{C_n + C_{n-1} + \cdots + C_0}$$

$$= \epsilon \left(1 - \frac{C_n + C_{n-1} + \cdots + C_{n-m}}{C_n + C_{n-1} + \cdots + C_0}\right), \ m > m_\epsilon = \textit{constant},$$

and hence $\lim\limits_{n=\infty} Y(n) = 0$. Thus the lemma becomes established.

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[*] This is also true when each of the quantities $C_0, C_1, \cdots, C_n$ is a function of n, other conditions remaining the same, and the proof which follows will be found to apply equally to such cases.

In particular, let us now take

$$C_n = \frac{r(r+1)\cdots(r+n-1)}{n!}, \quad r = \textit{integer} \geqq 1,$$

and place $c_n = s_n$, where s_n represents the sum of the first $n+1$ terms of (1). Then condition (4) is satisfied and (5) is at once seen to give the sum formula (2) of Cesàro. More generally, if V_n be any function of n such that the difference $V_n - V_{n-1}$ never decreases as n increases (n sufficiently large), while at the same time

$$\lim_{n=\infty} \frac{V_n - V_{n-1}}{V_n} = 0, \tag{6}$$

we may obtain in like manner a sum formula for the series (1) by taking $C_n = V_n - V_{n-1}$. In fact, we shall then have

$$\frac{C_n + C_{n-1} + \cdots + C_{n-m}}{C_n + C_{n-1} + \cdots + C_0} \leqq (m+1)\frac{V_n - V_{n-1}}{V_n},$$

so that condition (4) becomes satisfied, with which (5) yields the formula mentioned. In the special case leading to formula (2) it will be observed that

$$V_n = \frac{r(r+1)\cdots(r+n)}{n!}; \quad V_n - V_{n-1} = \frac{r(r+1)\cdots(r+n-1)}{n!}, \tag{7}$$

so that

$$\frac{V_n - V_{n-1}}{V_n} = \frac{r}{r+n}.$$

We turn our attention therefore to the function V_n defined by a general equation of the type

$$\frac{V_n - V_{n-1}}{V_n} A_n; \quad \lim_{n=\infty} A_n = 0. \tag{8}$$

A solution of this equation which reduces to (7) when

$$A_n = \frac{r}{r+n}$$

(case of Cesàro) is

$$V_n = \prod_{n=1}^{n}\left(\frac{1}{1-A_n}\right). \tag{9}$$

Let us now consider the function V_n obtained by taking

$$A_1 = \frac{1}{r+1}, \quad A_n = \frac{r}{r+\log n}. \qquad (n \geqq 2).$$

This function will necessarily satisfy (6) because of (8) and will be such that $V_n - V_{n-1}$ is an ever increasing function of n (n sufficiently large) since the result of increasing n by 1 is to multiply $V_n - V_{n-1}$ by

$$\frac{A_{n+1}}{A_n} \cdot \frac{V_{n+1}}{V_n} = \frac{r+\log n}{\log(n+1)} > \frac{\log(n+r)}{\log(n+1)} \geqq 1.$$

Thus we reach the sum formulas

$$(10) \quad s = \lim_{n=\infty} \frac{f_p(n, r)s_0 + f_p(n-1, r)s_1 + \cdots + f_p(0, r)s_n}{f_p(n, r) + f_p(n-1, r) + \cdots + f_p(0, r)},$$

where

$$f_p(n, r) = V_n - V_{n-1} = \frac{r(r+1)}{r+\log_p n} \prod_{n=2}^{n} \frac{r+\log_p n}{\log_p n} \quad (p = 0, 1)$$

with $\log_0 n = n$ (case of Cesàro) and $\log_1 n = \log n$.

More generally, by taking

$$A_1 = \frac{r}{r+1}, \quad A_n = \frac{r}{r+\log_p n},$$

where $\log_0 n = n$ and $\log_p n = \log\log \cdots \log n$ (p times) we are led to a function $f_p(n, r)$ defined by the formula

$$(11) \quad f_p(n, r) = \frac{r(r+1)}{r+\log_p n} \prod_{n=2}^{n} \frac{r+\log_p n}{\log_p n} \quad (p = 0, 1, 2, 3, \cdots),$$

which function satisfies conditions (a) and (b) of § 2.

In order to complete the proof of the theorem it remains therefore but to show that the function $f_p(n, r)$ defined by (11) satisfies (c) and (d).

Recalling that

$$\sum_{m=0}^{n} f_p(m, r) = \prod_{n=1}^{n} \left(\frac{1}{1-A_n} \right) = (r+1) \prod_{n=2}^{n} \frac{r+\log_p n}{\log_p n},$$

we have

$$(12)\quad \frac{\sum\limits_{m=0}^{n} f_p(m, r)}{\sum\limits_{m=0}^{n} f_p(m, r+1)} = \frac{r+1}{r+2} \prod_{n=2}^{n} \frac{r + \log_p n}{r + 1 + \log_p n},$$

$$= \frac{r+1}{r+2} \prod_{n=2}^{n} \left(1 - \frac{1}{r + 1 + \log_p n}\right),$$

and since the series

$$\sum_{n=2}^{\infty} \frac{1}{r + 1 + \log_p n}$$

is divergent, it follows from the well-known tests for convergence and divergence of infinite products that the last member of (12) approaches the limit 0 when $n = \infty$, with which condition (*c*) becomes satisfied.

Again we have

$$\frac{\sum\limits_{m=0}^{n} f_p(m, r)}{\sum\limits_{m=0}^{n} f_{p+1}(m, 1)} = \prod_{n=2}^{n} \frac{(r + \log_p n) \log_{p+1} n}{(1 + \log_{p+1} n) \log_p n} = \prod_{n=2}^{n} (1 - g_n),$$

where

$$g_n = \frac{\dfrac{1}{1 + \log_{p+1} n} - \dfrac{r}{r + \log_p n}}{1 - \dfrac{r}{r + \log_p n}}.$$

For all values of n we thus have

$$g_n > \frac{\log_p n \left(1 - \dfrac{r \log_{p+1} n}{\log_p n}\right)}{(r + \log_p n)(1 + \log_{p+1} n)} > \frac{c}{\log_{p+1} n}; \quad c = \textit{constant}$$

and hence the series

$$\sum_{n=2}^{\infty} g_n$$

is divergent, with which condition (*d*) becomes verified, thus completing the proof of the theorem.

ON FREDHOLM'S EQUATION.

BY PROFESSOR PAUL SAUREL.

(Read before the American Mathematical Society, April 24, 1909.)

THE formulas which Fredholm has given* for the solution of the integral equation

$$(1)\qquad \phi(x)+\lambda\int_0^1 \kappa(x, s)\,\phi(s)\,ds=\psi(x),$$

in which λ is a given constant, $\kappa(x, y)$ and $\psi(x)$ are given functions, while $\phi(x)$ is to be determined, are well known. Fredholm remarks that the theory of this equation may be considered as a limiting case of the theory of a set of linear algebraic equations, but in his published demonstration he makes no use of this remark and contents himself with a verification of his formulas. Hilbert† and Plemelj,‡ on the other hand, have obtained Fredholm's results by considering an infinite set of linear equations. More recently, Goursat§ and Lebesgue‖ have shown that for a kernel $\kappa(x, y)$ having the form

$$\kappa(x, y)=X_1(x)\,Y_1(y)+X_2(x)\,Y_2(y)+\cdots+X_n(x)\,Y_n(y),$$

one can easily obtain Fredholm's formulas. In the present note, I should like to show how the consideration of equation (1) leads directly to Fredholm's expressions.

If we set

$$(2)\qquad \phi(x)=\psi(x)-\rho(x),$$

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* "Sur une nouvelle méthode pour la résolution du problème de Dirichlet," *Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar*, Stockholm, vol. 57, p. 39 (1900). "Sur une classe de transformations rationnelles," *Comptes rendus*, vol. 134, p. 219 (1902). "Sur une classe d'équations fonctionnelles," *Acta Mathematica*, vol. 27, p. 365 (1903).

† "Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen," *Göttinger Nachrichten*, p. 49 (1904).

‡ "Zur Theorie der Fredholmschen Funktionalgleichung," *Monatshefte für Mathematik und Physik*, vol. 15, p. 93 (1904).

§ "Sur un cas élémentaire de l'équation de Fredholm," *Bulletin de la Société mathématique de France*, vol. 35, p. 163 (1907).

‖ "Sur la méthode de M. Goursat pour la résolution de l'équation de Fredholm," *Bulletin de la Société mathématique de France*, vol. 36, p 3 (1908).

equation (1) becomes

$$\rho(x) + \lambda \int_0^1 \kappa(x, s)\rho(s)\,ds = \lambda \int_0^1 \kappa(x, s)\,\psi(s)\,ds. \tag{3}$$

Before attempting to solve this equation in all its generality let us first examine the special case in which $\psi(x)$ is identically equal to zero. The resulting equation

$$\rho(x) + \lambda \int_0^1 \kappa(x, s)\rho(s)ds = 0 \tag{4}$$

is obviously satisfied by taking $\rho(x)$ identically equal to zero. But the analogy that exists between this equation and a set of homogeneous linear algebraic equations makes it natural to attempt to determine the expression whose vanishing is necessary in order that the equation may admit of a solution not identically equal to zero.

From equation (4) we get

$$\rho(x) = -\lambda \int_0^1 \kappa(x, s_1)\rho(s_1)ds_1. \tag{5}$$

If we add to each side of this equation the expression

$$\lambda\rho(x)\int_0^1 \kappa(s_1, s_1)ds_1,$$

the integrand on the right will take the form of a determinant and we get

$$\rho(x)\left[1 + \lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] = \lambda\int_0^1 \begin{vmatrix} \rho(x) & \kappa(x, s_1) \\ \rho(s_1) & \kappa(s_1, s_1) \end{vmatrix} ds_1 \tag{6}$$

$$= \lambda\int_0^1 \begin{vmatrix} \rho(x) & \kappa(x, s_2) \\ \rho(s_2) & \kappa(s_2, s_2) \end{vmatrix} ds_2. \tag{7}$$

In the second member of equation (7) let us replace $\rho(x)$, $\rho(s_2)$ by the values given by equation (5); we get

$$\begin{aligned} &\rho(x)\left[1 + \lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] \\ &\qquad = -\lambda^2\int_0^1\int_0^1 \rho(s_1)\begin{vmatrix} \kappa(x, s_1) & \kappa(x, s_2) \\ \kappa(s_2, s_1) & \kappa(s_2, s_2) \end{vmatrix} ds_1 ds_2. \end{aligned} \tag{8}$$

An interchange of the subscripts 1, 2 in the expression on the right of this equation will not change its value; we may therefore write

$$
(9)\quad \rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right]=\frac{\lambda^2}{2}\int_0^1\int_0^1\left[-\rho(s_1)\begin{vmatrix}\kappa(x, s_1) & \kappa(x, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix}\right.
$$

$$
\left. -\rho(s_2)\begin{vmatrix}\kappa(x, s_2) & \kappa(x, s_1)\\ \kappa(s_1, s_2) & \kappa(s_1, s_1)\end{vmatrix}\right] ds_1 ds_2.
$$

On adding

$$
\frac{\lambda^2}{2}\rho(x)\int_0^1\int_0^1\begin{vmatrix}\kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix} ds_1 ds_2
$$

to each side of this equation, the integrand on the right will take the form of a determinant and we get

$$
\rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1+\frac{\lambda^2}{2}\int_0^1\int_0^1\begin{vmatrix}\kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix} ds_1 ds_2\right]
$$

$$
(10)\qquad =\frac{\lambda^2}{2}\int_0^1\int_0^1\begin{vmatrix}\rho(x) & \kappa(x, s_1) & \kappa(x, s_2)\\ \rho(s_1) & \kappa(s_1, s_1) & \kappa(s_1, s_2)\\ \rho(s_2) & \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix} ds_1 ds_2.
$$

Equations (6) and (10) suggest at once the general formula

$$
\rho(x)\left[1+\sum_1^n\frac{\lambda^p}{p!}\int_0^1\int_0^1\cdots\int_0^1\kappa\begin{pmatrix}s_1, s_2, \cdots, s_p\\ s_1, s_2, \cdots, s_p\end{pmatrix}ds_1 ds_2\cdots ds_p\right]
$$

$$
(11)\qquad =\frac{\lambda^n}{n!}\int_0^1\int_0^1\cdots\int_0^1\begin{vmatrix}\rho(x) & \kappa(x, s_1) & \cdots & \kappa(x, s_n)\\ \rho(s_1) & \kappa(s_1, s_1) & \cdots & \kappa(s_1, s_n)\\ \vdots & \vdots & & \vdots\\ \rho(s_n) & \kappa(s_n, s_1) & \cdots & \kappa(s_n, s_n)\end{vmatrix} ds_1 ds_2\cdots ds_n,
$$

in which

$$
\kappa\begin{pmatrix}s_1, s_2, \cdots, s_p\\ s_1, s_2, \cdots, s_p\end{pmatrix}
$$

is used to denote the determinant of the pth order in which the element in the ith row and jth column is $\kappa(s_i, s_j)$.

It is not hard to establish equation (11) by induction. Moreover, by using Hadamard's theorem * that the modulus of a determinant of the nth order is less than $M^n\sqrt{n^n}$, M being a number greater than the modulus of every element, it is possible to show that as n increases without limit the right side of equation (11) approaches zero as a limit. Accordingly it follows that in order that equation (4) may have a solution not identically zero it is necessary that

$$D(\lambda) = 0,$$

where

$$(12)\quad D(\lambda)=1+\sum_1^\infty \frac{\lambda^p}{p!}\int_0^1\int_0^1\cdots\int_0^1 \kappa\begin{pmatrix} s_1, & s_2, & \cdots, & s_p \\ s_1, & s_2, & \cdots, & s_p\end{pmatrix} ds_1 ds_2 \cdots ds_p.$$

As is well known, the convergence of (12) for every value of λ can be established by means of Hadamard's theorem.

Let us now return to equation (3). Guided by the analogy which exists between this equation and a set of non-homogeneous linear equations we may assume that $\rho(x)$ is equal to a fraction of which $D(\lambda)$ is the denominator. Accordingly, if we set

$$\rho(x) = \frac{\sum_0^\infty \lambda^n A_n(x)}{D(\lambda)},$$

and if we substitute in equation (3) we shall find without much difficulty the values of the functions $A_n(x)$; we shall thus obtain Fredholm's formulas.

We can reach the same result more simply, however, by treating equation (3) as we have treated equation (4). From equation (3) we get

$$(13)\quad \rho(x) = -\lambda\int_0^1 \kappa(x, s_1)\rho(s_1)ds_1 + \lambda\int_0^1 \kappa(x, s)\psi(s)ds.$$

As before, we add

$$\lambda\rho(x)\int_0^1 \kappa(s_1, s_1)ds_1$$

to each side of this equation and we obtain an equation analogous to equation (7)

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* "Résolution d'une question relative aux déterminants." *Bulletin des Sciences Mathématiques*, 2d series, vol. 17, p. 240 (1893). Cf. W. Wirtinger, "Sur le théorème de M. Hadamard relatif aux déterminants," ibid., 2d series, vol. 31, p. 175 (1907).

$$\rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] = \lambda\int_0^1 \begin{vmatrix}\rho(x) & \kappa(x, s_2)\\ \rho(s_2) & \kappa(s_2, s_2)\end{vmatrix} ds_2 + \lambda\int_0^1 \kappa(x, s)\psi(s)ds. \tag{14}$$

In the second member of this equation let us replace $\rho(x)$, $\rho(s_2)$ by the values given by equation (13). We thus get an equation analogous to equation (8)

$$\begin{aligned}\rho(x)\left[1+\lambda\int_0^1 \kappa(s_1, s_1)ds_1\right] &= -\lambda^2\int_0^1\int_0^1 \rho(s_1)\begin{vmatrix}\kappa(x, s_1) & \kappa(x, s_2)\\ \kappa(s_2, s_1) & \kappa(s_2, s_2)\end{vmatrix} ds_1 ds_2\\ &\quad + \lambda\int_0^1 \kappa(x, s)\psi(s)ds\\ &\quad + \lambda^2\int_0^1\int_0^1 \begin{vmatrix}\kappa(x, s) & \kappa(x, s_1)\\ \kappa(s_1, s) & \kappa(s_1, s_1)\end{vmatrix}\psi(s)ds_1 ds.\end{aligned} \tag{15}$$

If we treat this equation as we treated equation (8), it is clear that we shall obtain an equation like equation (10) but differing from it by the presence of the two additional terms

$$\lambda\int_0^1 \kappa(x, s)\psi(s)ds + \lambda^2\int_0^1\int_0^1 \begin{vmatrix}\kappa(x, s) & \kappa(x, s_1)\\ \kappa(s_1, s) & \kappa(s_1, s_1)\end{vmatrix}\psi(s)ds_1 ds$$

on the right. Continuing in this way it is not hard to see that we shall get instead of equation (11) an equation differing from it by the presence on the right of the expression

$$\int_0^1 \psi(s)F_n(x, s, \lambda)ds \tag{16}$$

where

$$\begin{aligned}F_n(x, y, \lambda) &= \lambda\kappa(x, y)\\ &\quad + \sum_1^{n-1}\frac{\lambda^{p+1}}{p\,!}\int_0^1\int_0^1\cdots\int_0^1 \kappa\begin{pmatrix}x, & s_1, & \cdots, & s_p\\ y, & s_1, & \cdots, & s_p\end{pmatrix} ds_1 ds_2\cdots ds_p.\end{aligned} \tag{17}$$

If, in this new equation, analogous to (11), we allow n to increase without limit we obtain

$$\rho(x)D(\lambda)=\int_0^1 \psi(s)F(x, s, \lambda)ds, \tag{18}$$

where $F(x, y, \lambda)$ denotes the limit to which $F_n(x, y, \lambda)$ tends,

$$F(x, y, \lambda)=\lambda\kappa(x, y)$$

$$+\sum_1^\infty \frac{\lambda^{p+1}}{p!}\int_0^1\int_0^1\cdots\int_0^1 \kappa\begin{pmatrix} x, & s_1, & \cdots, & s_p \\ y, & s_1, & \cdots, & s_p\end{pmatrix} ds_1 ds_2 \cdots ds_p. \tag{19}$$

The convergence of this expression can be established by Hadamard's theorem.

If, in equation (18), we replace $\rho(x)$ by its value taken from equation (2), we get Fredholm's solution of equation (1)

$$\phi(x)=\psi(x)-\frac{1}{D(\lambda)}\int_0^1 F(x, s, \lambda)\psi(s)\,ds. \tag{20}$$

It is to be observed that the above demonstration establishes the uniqueness of the solution for every value of λ for which $D(\lambda) \neq 0$.

NEW YORK,
April 1, 1909.

THE CHICAGO SYMPOSIUM ON MATHEMATICS FOR ENGINEERING STUDENTS.

Symposium on Mathematics for Engineering Students. Being the Proceedings of the Joint Sessions of the Chicago Section of the American Mathematical Society and Section A, Mathematics, and Section D, Mechanical Science and Engineering, of the American Association for the Advancement of Science, held at the University of Chicago, December 30 and 31, 1907.

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ENGINEERING education, having passed through an infancy somewhat of the "ugly duckling" type in the academic family, and then an adolescence of growth too rapid for either garments or comfort, seems now to have arrived at a stage of maturity in which it recognizes the necessity for heart-searching as to its own real character and habits of life. The demands upon its energies from all quarters are heavy and insistent. It must maintain its vigor and efficiency by ridding itself of every superfluous ounce of adipose, and by keeping brain, nerves, and muscles in the best condition. Thus the traditional fundamentals of diet and hygiene must be critically scrutinized, and first of all, mathematics.

Mathematics from its antiquity and its necessity has so long been cast into pedagogic moulds that most people cannot separate their notions of it from these moulds — arithmetic, geometry, trigonometry, etc. It has been taught in some form to nearly all civilized persons, and unfortunately, but necessarily, to a great extent by teachers of very slender mathematical endowment — for this endowment is exceedingly rare. Its form and subject matter have been determined by all sorts of considerations of a more or less arbitrary character.

In our own country most of those who took college work in mathematics more than, say, twenty years ago — and this naturally includes most of the participants in the present symposium — were diligently drilled in Todhunter or some dilution of his texts, with little chance of appreciating the real vitality of mathematics or its relations to other living subjects. Those who escaped permanent aversion for it, had by their industry acquired mathematical technique and some power of concentration. This sort of education, while sometimes redeemed by the teacher's personality and doubtless not inferior to much contemporary work in other lines, could not long withstand the influence, on the one hand, of the new generation of mathematicians "made in Germany," and on the other, of competition with the modern scientific or technological subjects. It is safe to say that it hardly now survives to any important extent — except in the minds of certain critics who still describe the conditions of their own youth with fine but unscientific indifference to the progress made in the intervening years. On the other hand, it is but just to grant that the critical revision of a great body of knowledge, and the training of an army of teachers with new points of view — both indispensable steps — have not been matters for a day or a decade. The mere substitution of engineers to teach mathematics would have been vastly simpler, but of course utterly inadequate, not to say futile, as a remedy.

In the accomplishment of this piece of educational evolution, cooperation has been, and will continue to be, essential, and it is as a welcome form of such cooperation that the Chicago symposium, which is here reported, has its chief significance.

The report opens with a valuable description of the "present condition of mathematical instruction for engineers in American colleges," based on a study of seventeen selected institutions, by Professor Townsend, of Illinois. There is a judicious deprecation of undue advance of entrance requirements in mathematics, and of undue emphasis on uniformity. The statistical tables of entrance requirements and of time allotment in course are naturally to be used only with caution for purposes of comparison, and with due regard for Poincaré's dictum, that mathematics — a fortiori statistics — is the art of calling different things by the same name. Interesting references are made to the desirability of elective mathematics, to the positive and negative value of the "Perry movement," to recent tendencies to deal with mathematics as one subject rather than as many, etc. It is wisely urged that what is best for the engineering student in the first two years of his mathematical work is also best for the student who is taking mathematics as an element in a liberal education. In view of the emphasis put by some other speakers on the use of mathematics as a "tool," it may be pertinent to supplement the remark just quoted by adding that the best engineering schools and departments aim to combine in a truly liberal sense general and professional education, and that any attempt to teach mathematics merely as a tool would, to the extent of its dubious success, mean sacrificing the first of these objects. One of the important elements in higher — as distinguished from merely technical — education is surely the student's relation with teachers of widely different points of view, mathematical, philosophical, literary, etc., as well as scientific and engineering. It is perhaps natural that the same critics who see in mathematics nothing but a machine into which data are fed, and from which results are ground out, should most insist that the subject ought to be taught as a tool.

The symposium includes further a paper on conditions in foreign countries (chiefly Germany) by Professor Ziwet of Michigan, general papers by Mr. C. F. Scott of the Westinghouse Company and Dr. R. S. Woodward of the Carnegie Institution, a presentation of the standpoint of the practicing engineer by Mr. Ralph Modjeski of Chicago, and Mr. J. A. L. Waddell of Kansas City; of the standpoint of the professor of engineering by Professors Williams of Michigan, Talbot of Illinois, and Swain of the Massachusetts Institute of Technology; of the standpoint of the professor of mathematics, by Professors Slichter of Wisconsin, and Woods of the Massachusetts Institute of Technology, and by President McNair of the Michigan College of Mines.

For the readers of this BULLETIN it may naturally be most profitable to consider the criticisms of the engineers—including Mr. Scott—and the professors of engineering. Of the three engineers, one who received his mathematical training abroad and naturally expresses himself with some reserve in regard to American conditions, looks forward to the time when only the applied mathematics will be taught in college, all necessary abstract mathematics forming a part of the entrance requirements. The other two dwell on unsatisfactory elements in existing conditions, and on results to be sought, both of them emphasizing the importance of skill in the use of mathematics as a tool. This view, urged still more emphatically by one of the professors of engineering, might remind an unsympathetic reader of the ingenuous evidence so often presented to the Committee on Ways and Means by beneficiaries of the protective tariff. As a matter of course, the mere use of mathematics as a tool can not be most successfully taught by mathematical specialists. Exactly this is, and must always be, one of the most important functions and opportunities of the teachers of physics, mechanics, and engineering, as was naturally insisted in the course of the discussion by the mathematical speakers. The mathematicians must build a foundation adapted to the educational superstructure. With this their responsibility ends. If they permit themselves to be diverted from this responsibility for laying a solid foundation, it needs no engineer to foretell the disastrous consequences.

The further criticism of the engineers and the professors of engineering, while varying widely in details, seems to connect itself with the general notion that mathematics is habitually presented to engineering students in too abstract a form, with too little attention to the statement of problems and the interpretation of results, and with too little appreciation of the uses to be made of mathematics in engineering work. There is still unquestionably a measure of truth in this criticism, but for reasons which have been already indicated, the extent to which the criticism is valid can be determined for each institution only by a careful investigation of what it is now doing. No one of the speakers in question mentions having made such an investigation, even in a single institution. One of the engineers remarks that "possibly methods have changed of late years, but nothing that the writer has seen or heard indicates to him that any fundamental improvement has been effected." One of the professors of engineering states that in his classes he has had students from most of the principal colleges and technical schools of the country, and that he has failed to notice any great difference in them in respect to lack of mathematical power. Such statements hardly suggest that degree of careful discrimination which may be fairly expected of a scientific critic on a scientific subject.

The criticism that teachers of mathematics neglect the applications, so that their students do not learn to *do* anything, suggests the inquiry: Just what do they need to learn to do? In the large engineering institutions and departments the mathematical instruction must often be given to classes made up of students from several courses, or departments, for whom the number and variety of applications of common interest will naturally be limited. Moreover, the stronger engineering schools draw a considerable — and probably an increasing — proportion of their students from academic institutions, in which mathematics has been taught necessarily without specialization. Under these conditions all that can fairly be demanded of the teachers of mathematics is that they articulate their work as closely as may be with the dependent physics and mechanics which immediately succeed it, by including physical and mechanical problems of general interest.

The plea of several speakers that geometry be more emphasized, and that descriptive geometry made more mathematical, deserves careful attention. This question is, of course, to a considerable extent one of organization rather than of mathematical teaching per se. Any thorough-going improvement would seem to imply that descriptive geometry, and perhaps mechanical drawing, should be given under the direction of the mathematical department as its laboratory work, rather than independently, as is now apt to be the case. Undoubtedly teachers of mathematics would welcome any tendency on the part of their colleagues having charge of these outlying subjects to give them a more mathematical character.

It is not without interest that the most severe criticism of the present teaching and teachers of mathematics is coupled with a citation of Sir William Hamilton as an authority on the value and the pedagogy of mathematics. In view of the vogue of Hamilton's criticism, and the limited publicity of Professor Keyser's recent exposé of it, a brief quotation may be made from his Columbia University lecture on Mathematics. In Hamilton's article "the reader is apparently confronted with the assembled opinions of the learned world, and — what is more amazing — they all agree. Literati of every kind, of all nations and every tongue, orators, philosophers, educators, scientific men, ancient and modern, known and unknown, all are made to support Hamilton's claim, and even the most celebrated mathematicians seem eager to declare that the study of mathematics is unworthy of genius and injures the mind. . . . The Scotchman's victory was complete, his fame enhanced, and his alleged judgment regarding a great human interest of which he was ignorant has reigned over the minds of thousands of men who have been either willing or constrained to depend on borrowed estimates. But even all this may be condoned. Jealousy, vanity, parade of learning, may be pardoned even in a philosopher. Hamilton's deadly sin was none of these, it was sinning against the light." It has been shown by Bledsoe and Pringsheim "that Hamilton by studied selections and omissions deliberately and maliciously misrepresented the great authors from whom he quoted — d'Alembert, Blaise Pascal, Descartes and others — distorting their express and unmistakable meaning even to the extent of complete inversion." Even a philosopher thus discredited may have told some truths of his own, but his opinion can hardly be deemed to add much weight to the present criticism. Any attempt to discuss the general educational value of mathematics would, however, far exceed the limits of this review, besides being presumably unnecessary for its readers.

It should not be inferred from any thing that has been said that adverse criticism of mathematics, or mathematicians, was general on the part of the engineering speakers. It is too often the extreme criticism — picturesque rather than accurate — which impresses even a professional public, but the symposium as a whole was rather notable for agreement in fundamentals. As to the importance of the teacher's training and continued interest in applied mathematics, as to the need of emphasis on concrete problems, as to the relatively low value of lectures and mathematical blackboard work, there appears to have been no difference of opinion. It may be added, in connection with the training of teachers of mathematics, that there would be many advantages if a larger share in that training could be taken by the technological schools and departments than is at present the case.

The case for the teachers of mathematics was ably sustained by Professors Slichter and Woods. The former from the standpoint of a consulting engineer as well as that of a teacher of mathematics, emphasizes the increasing need on the part of technology for more and better science, and insists that mathematics is a basal science rather than a tool. Professor Woods dwells particularly on the urgent need that teachers in various subjects dependent on mathematics continue and complete the work which the mathematicians can only have begun. Mention may also be made of the valuable contribution to the subsequent discussion by President Howe of the Case School.

The pamphlet closes with the announcement of a committee of fifteen — the actual list seems to include twenty names — to make to the Society for the Promotion of Engineering Education such a report on mathematics for colleges of engineering as in their opinion will be of service to teachers in such institutions.

After all, however, the real value and importance of such a symposium consists in the intangible influence of the discussion itself on its hearers and readers. Reports and resolutions are at the best merely the stamp on the coin, and too often make but a faint impression thereon. The teaching of mathematics to students of engineering is still in that stage of development by individual initiative in which diffusion of information and circulation of ideas are invaluable, but in which ideas formulated in one year as novel may in the next seem commonplace. Up to a certain point it is easy to agree as to principles; beyond that point efforts at agreement may be for the time quite futile. What is difficult, yet possible and necessary, is a constant improvement of our individual practice. It would be a great service — but an exceedingly difficult one — for any committee or any individual to bring the criticisms of current mathematical teaching to the test of a searching statistical investigation, the only conclusive test. It is also a great but a quite practicable service to prepare for the use of teachers and students of mathematics texts involving more numerous and more varied applications. In this direction the teachers of mathematics have done much and need all support and cooperation — criticism also, if it be constructive, or, at least, discriminating. The need of improvement in line with the constructive criticism is, in the judgment of the present reviewer, freely recognized and continually better met.

H. W. TYLER.

OSGOOD'S CALCULUS.

A First Course in the Differential and Integral Calculus. By WILLIAM F. OSGOOD, PH.D., Professor of Mathematics in Harvard University. New York, the Macmillan Company, 1907, pp. xv + 423. Revised edition, 1909, pp. xv + 462.

PROFESSOR OSGOOD in his presidential address before the AMERICAN MATHEMATICAL SOCIETY * has discussed and illustrated the principles which his experience has led him to consider should govern the teaching of the calculus. In the present text he gives us the detailed application of those principles to the difficult pedagogical problems which confront the instructor in the first course in this subject.

Successful instruction in mathematics requires careful adjustment of the conflicting claims of rigor, formalism, and interest. Professor Osgood has recognized † that rigor is a relative matter particularly in elementary instruction, and has enunciated the principle that in such instruction a discussion is to be regarded as rigorous if it meets all the logical demands which the student can be regarded as capable of appreciating at that time. This principle is at bottom the same as that which governs contemporary judgment of productive work, and its application to instruction is but a recognition of the fact that the mathematical development of the individual differs in general from that of the race at most by a transformation of similarity.‡ It is evident, however, that such a principle must be applied with care, for otherwise it may be cited in defense of a multitude of mathematical sins. If used with judgment, however, as is the case in this text, it becomes the very foundation of successful mathematical teaching.

The applications of this principle are in evidence throughout the book. For example: the theorem on the limit of the sum is at first (page 12) tacitly assumed, then (page 14) mentioned in a footnote, and finally proved (page 15) when the progress of the work has rendered evident its necessity. Again in the discussion of $D_x(u+v) = D_xu + D_xv$ and similar theorems (pages 14, 21) the existence of D_xu and D_xv is assumed without comment, but later (page 423) the assumption and its significance are discussed. A similar point is illustrated in the treatment of $D_xy = D_ty/D_tx$ (pages 23, 423).

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* BULLETIN, vol. 13 (1907), pp. 449–467.

† *Annals of Mathematics*, ser. 2, vol. 4 (1903), p. 178.

‡ Cf. Cantor, Geschichte der Mathematik, 3 Aufl., Bd. 1, p. 3.

It is in conformity with this principle too, that Professor Osgood, an authority on questions of uniform convergence, has omitted this important subject from his text. The omission is at first sight somewhat startling in view of the author's tendencies, and yet when one reflects that the genius of a Weierstrass was required to establish uniform convergence as a constructive principle, the wisdom of omitting the subject from a course for beginners is apparent. As a consequence of this omission the continuity, differentiation, and integration of series are merely mentioned with a reference to the author's Infinite Series.

Progressive criticism of results already obtained and of methods already used must form an important part of a system of instruction which continually modifies its standard of rigor to accord with the student's increasing power of comprehension. This criticism is the more valuable to the student in proportion as it is constructive rather than destructive. It is therefore desirable that the necessarily incomplete discussions of an elementary course should be susceptible of ready modification in order to satisfy later more exacting demands, but they should not be such as to require extensive reconstruction. A case in point here is the treatment of Duhamel's theorem (page 164). The theorem is given in the original inexact form in which the question of uniformity is so veiled that in order to bring out the precise point at issue considerable destructive criticism of the type which Professor Osgood himself has given * is rendered necessary. Professor Osgood's own revised formulation † is as simple in statement and proof as the original, if the question of uniformity is waived, and it has the important advantage of needing only to be supplemented rather than reconstructed in a later and more advanced course. The revised form has, moreover, the further advantage of more direct application to the problem wherein the theorem is most valuable, namely, that of the formulation of questions involving the definite integral as the limit of a sum.

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* Osgood, *Annals of Mathematics*, ser. 2, vol. 5 (1903), p. 169ff.

† Ibid., p. 173

The formal and manipulative side of the calculus has till recently received principal emphasis in American texts, and indeed to such an extent that the student has too frequently failed to obtain an adequate conception of the fundamental ideas of the subject and their application to concrete problems.

On these fundamental ideas and methods rather than on mere manipulation the stress is laid in Professor Osgood's text. For example an extended graphical and analytical treatment of $D_x x^n = nx^{n-1}$ (n fractional and negative) is given; the existence of $e = \lim_{n=\infty} (1 + 1/n)^n$ is accurately though simply established, and the idea of the definite integral as the limit of a sum is developed with careful attention to detail by means of a series of geometrical and statical problems. On the other hand in place of the usual treatment of the integration of rational fractions, of $\int R(x, \sqrt{ax^2 + bx + c})dx$, and of $\int x^m(a + bx^k)^p dx$, there is a short introduction to the use of the well known integral tables of B. O. Peirce. The most serious objection to this departure from the conventional treatment lies in the fact that indefinite integration, as an inverse operation, is always a difficult process for the student, and the power to reduce a given function to a known integrable form, *e. g.*, to one of those in the tables, is in general gained only by considerable directed practice. The traditional discussion of the types above mentioned may well be displaced by an introduction to the use of the tables, but in the opinion of the reviewer that introduction should contain a somewhat detailed discussion of the simpler integrable forms; a statement concerning forms not in general integrable, *e. g.*, $\int R(x, \sqrt{ax^3 + 3bx^2 + 3cx + d})dx$; and a statement at least of the theorems concerning the integrability of $R(x)$, $R(x, \sqrt{ax^2 + bx + c})$, $R(\sin x, \cos x)$, etc.

Abundant opportunity for practice and differentiation and integration is afforded by the supplementary exercises (revised edition, pages 324–359).

The space and time saved in the chapter on integration has been used to most excellent advantage in that on partial differentiation.

In much of the work of the elementary course in the calculus the analytical difficulties are those arising in the proofs of the theorems used. Here the trouble is almost invariably one of interpretation. The meaning of the partial derivative especially when several dependent and independent variables occur is seldom understood. The important section (page 306) on the notation used for partial derivatives, and the problems 4, 8, 21, 24, pages 312–315 form a much needed addition to the usual treatment of this subject, and will be a welcome aid to the teacher and student of thermodynamics. Were such discussions more common in texts on the calculus, their proper place, they would be unnecessary in treatises on thermodynamics.* The treatment of the various cases of $du = \partial u/\partial x\, dx + \partial u/\partial y\, dy$, etc. (page 292ff.), is another valuable aid to the clarification of the student's ideas concerning partial differentiation, and together with the treatment of the differential of a function of a single variable (page 92ff.) applies the principles concerning differentials enunciated in the author's presidential address † in such a way as to leave no opportunity for the "little zeros" to insinuate themselves into the student's mind.

Professor Osgood has long emphasized by precept and example the importance of developing new mathematical concepts in the student's mind by means of problems, *i. e.*, of causing the new mathematical idea to appear as a necessary element for the solution of a definite geometrical or physical problem.

His chapter on the definite integral as the limit of the sum is an example of this method, as well as of gradual development of precision in concept and demonstration conformably to the student's advance in assimilative power. The important purely theoretical matters in such a chapter are the concept of the definite integral as the limit of a sum, the upper and lower integrals, the fundamental theorem of the integral calculus, the proof of the existence of the definite integral, and the theorem of Duhamel. In Professor Osgood's treatment the area problem leads (page 153) to the concept of the definite integral as the analytic formulation of a limit the existence of which is geometrically evident. A previous discussion (pages 111–2) had developed the relation $d(\text{area})/dx = \text{ordinate}$. Comparison gives the fundamental theorem of the integral calculus. Of course the existence of an area is tacitly assumed, but that is at this stage of minor importance. The essential thing is that the student visualize the definite integral and understand its relation to the anti-derivative, *i. e.*, that he recognize the first as formulating his problem, the second as a means of evaluating its result, and shall see the necessity of the fundamental theorem of the integral calculus as the link which connects his method of formulation with his method of evaluation.

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* E. g., Bryan, Thermodynamics, Chapter III, pp. 21–26.

† BULLETIN, loc. cit., p. 451.

The next step is the introduction of the upper and lower integrals.* The problem of the fluid pressure upon a vertical plane surface with curvilinear boundary leads to the establishment of the element of the upper and lower integrals, their relation to the element of the integral and the passage to the limit which shows the existence of the upper and lower integrals, their equality and the existence of the integral. Here the reasoning depends on geometrical and statical considerations.

The way is now prepared for the analytic proof of the existence theorem for the definite integral. The proof is given in the usual way for a monotonic function. Though it is illustrated by the figure of the circumscribed and inscribed areas, and the language is in part geometrical, no fact is used the arithmetic truth of which is not immediately evident from the definition of the monotonic function. The extension to the case of a finite number of maxima and minima is then made, and finally in the appendix (page 423) attention is called to the case of functions with an infinite number of maxima and minima.

The chapter is noteworthy for the manner in which the difficulties of the theory of the definite integral are brought to the student's attention as inherent in the problems to which the integral calculus is applicable.

The chapters on double and triple integrals follow the same lines and there is the same gradual development of the fundamental concepts and avoidance of unnecessary analytical difficulties. The insistence on the distinction between the double (or triple) integral and the iterated integral, and on the fundamental theorem of the integral calculus as the link which joins them, is as desirable as it is unusual in elementary texts.

Few students, even those for whom mathematics is an elective rather than a required subject find their chief interest in the precision of its concepts, the rigor of its demonstrations or the elegance of its formulas. They continually demand immediate application to concrete problems as a reason for considering the subject at all.

Professor Osgood's treatment is well fitted to satisfy this demand. For example, as soon as the derivative has been defined and a few rules of differentiation established, the new method is applied to problems of maxima and minima, of velocity, of curve tracing, and of roots of equations in such a way as to make evident its power as compared with that of elementary methods. The result is that the first three chapters would by themselves form an excellent introduction to the differential calculus for those students who desire a knowledge of its scope and methods without going into analytical details.

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* The terms upper and lower integrals are not mentioned, but the concept is used.

The earlier American texts in the calculus have in general confined their applications to the field of geometry. The adherents of the "Perry movement" have gone to the other extreme and have produced a flood of problems taken from engineering practice. They have, however, too often forgotten that the student for whom the problems are designed has not yet acquired the technical knowledge necessary for an appreciation of their meaning and importance.

Professor Osgood has by no means neglected geometry, as his chapters on the cycloid, on curvature and evolutes, on envelopes, on partial differentiation, and on definite integrals bear witness. On the other hand, he has throughout the text, and especially in the chapter on mechanics, provided numerous problems illustrating the application of the calculus to physical phenomena, and that without requiring more technical knowledge than can be introduced into the text without distracting the student's attention from the true aim of the work — the application of the calculus to physical phenomena, as opposed to the formulation of a particular physical problem.

In conclusion it may be said that Professor Osgood's text is characterized by insistence on the real essentials of the calculus, and by consistent maintenance of that close relation between theory and application to which both pure and applied mathematics owe their most important advances.

CHARLES N. HASKINS.

THE UNIVERSITY OF ILLINOIS,
March, 1909.

SHORTER NOTICES.

Grundlehren der neueren Zahlentheorie. By PAUL BACHMANN. Leipzig, G. J. Göschen, 1907. Sammlung Schubert, v. 53. 11 + 270 pp.

KLEIN has recently * called attention to the two fundamental types of mathematical development. The one is concerned primarily with the exposition of a given branch of mathematics for its own sake and with its own methods. If this type alone obtained, mathematics would appear as a series of more or less distinct theories — algebra, trigonometry, calculus, etc. — which may show here and there incidental points of contact, but which are not organically connected. The other type of development, on the other hand, is concerned with just this welding together of the various so-called branches into a unified whole. It conceives, for example, the two great divisions of analysis and geometry as being only two different aspects of the same thing. Wherever this second type of development has made itself prominently felt it has meant a gain in power; not simply by giving various interpretations to the same theorem, but, and perhaps chiefly, by making the methods of one theory available for research in another.

All branches of mathematics have felt the influence of this second type of development to a greater or less degree. Even the theory of numbers, which for a while held a certain autonomy in its methods, is at present in its more advanced portions at least in intimate union with other fields, *e. g.*, the theory of functions of a complex variable. In recent years, however, even the elementary portions of the theory of numbers have had to submit to the introduction of matter and methods from another field; viz., that of geometry. To speak of a "modern" treatment of the elementary theory of numbers brings to mind at once the lectures of Klein — that master exponent of the second type of development — during the year 1895–96, and Minkowski's work on the Geometrie der Zahlen. An elementary text on the theory of numbers which makes consistent use of these geometric interpretations and methods where they are available is obviously a great desideratum. It was therefore with much joyful anticipation that the reviewer opened this recent addition to the Sammlung Schubert. The preface seemed to intimate that the author intended to write just such a text as we had hoped for. But after reading the little book with much occasional pleasure the reviewer laid it down with a distinct feeling of disappointment. Of geometric interpretation there is a little; of geometric method there is none.

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* Elementarmathematik vom höheren Standpunkte aus, pp. 180–187.

Having thus expressed our disappointment regarding one feature of the text, it is a pleasure to mention some that are pleasing. The notion of a domain of rationality — or field — is introduced on the first page, and that of a modulus — domain of integrality — on the second. The latter is defined as any set of numbers which is closed under the operations of addition and subtraction; *i. e.*, such that if a and b are numbers of the set, $a + b$ and $a - b$ are also in the set. Thus, if $a, b, c, \ldots$ are any given numbers, the set of all numbers of the form $ax + by + cz + \ldots$, where x, y, z, are any integers, forms a modulus which is denoted by $[a, b, c, \ldots]$. This notion is made fundamental throughout the book; a fact which differentiates it from the ordinary treatment, and which introduces a refreshing unity of conception. Many results are stated in terms of it which are usually stated differently. For example, two integers n, n' are said to be congruent mod m if $n - n'$ is a number of $[m]$. Its advantageous use in the development of Dedekind's theory of ideals is obvious.

The book is divided into two parts devoted respectively to "the rational number field" (157 pages) and to "the quadratic number field" (113 pages). The first four chapters discuss briefly and sometimes with a refreshing novelty the usual topics relating to 1) the divisibility of numbers, 2) congruences, 3) quadratic residues, including the law of reciprocity, 4) the linear form $f = ax + by$. Another unifying principle is found in the fact that the theorem concerning the solvability in integers of the equation $ax + by = 1$ is made fundamental throughout this portion of the text. Chapter 4 contains a complete discussion of the equivalence of moduli $[a, b]$, which is of fundamental importance later in the discussion of ideals. Here also we have for the first time a geometric interpretation of the results by means of a lattice (Zahlengitter). We are sorry that in this chapter the author did not see fit to include also Klein's elegant geometric interpretation of the development of an irrational number into a continued fraction. Chapter 5, which is the last of Part 1, begins the discussion of the equivalence of quadratic forms, the discussion not being completed until later in connection with the theory of quadratic numbers.

The author has developed the theories of quadratic forms and quadratic numbers in conjunction in Part 2, each supplementing the other and forming together a single theory. In this fact he finds his chief reason for calling his treatment "modern." The result is indeed "esthetically satisfying" (quotation from the preface). In view of the fact, however, that the book is professedly intended for beginners, it may perhaps be doubted whether this treatment is pedagogically desirable. There can be no manner of doubt that the beginner will find Part 2 hard reading; and it does not appear evident that the intermingling of forms and ideals, however beautiful the result, makes the reading any less difficult.

The book is remarkably free from typographical errors. Besides the single one noted in the corrigenda, the reviewer has noticed only two; one on page 36, 3d line from below, where *nr*. should read Nr.; and one on page 70, line 7, where the reference should be to Nr. 4 instead of to Nr. 3. In this connection we may note further that on page 25 the expression $\phi(1)$ must be defined as equal to unity, and on page 71 the condition $\Delta \neq 0$ should be added.

J. W. YOUNG.

The Axioms of Descriptive Geometry. By A. N. WHITEHEAD. Cambridge University Press, 1907. viii + 74 pp.

THIS little volume is No. 5 of the Cambridge Tracts in Mathematics and Mathematical Physics. It follows a previous tract (No. 4 of the same series) by the same author, on the Axioms of Projective Geometry, to which constant reference is made. The work begins with formulations of the axioms, those of Peano and Veblen being given in detail. Chapter II treats of the relation of projective space and the associated space obtained by taking a convex region of the projective space, such a convex region being shown to be a descriptive space. The development follows that of Bonola closely. Chapter III contains the development of ideal elements in a descriptive geometry, the work being drawn from Veblen, apparently. In Chapter IV a "General theory of correspondence" is introduced through the medium of projective coordinates, the ideas of continuous groups of projective transformations and their infinitesimal transformations being developed from analytic geometry. Chapters V–VII contain a logical analysis of the method of superposition as applied to geometrical proofs. After the axioms of congruence of Pasch, the development of congruence in terms of previous concepts is given, after Lie, by consideration of congruence groups and their subgroups. Metrical geometries are then treated with reference to the absolute, and the kind of a metrical geometry obtained from the congruence group is shown to depend upon an assumption concerning the nature of the absolute.

The proofs are usually only sketched and the reader is left to complete the details. The work could be improved by a more formal statement of the theorems and the exact group of axioms upon which each theorem depends. For the reader who has some knowledge of the general methods used and the point of view, the book should serve as a very useful compilation of results from many sources.

F. W. OWENS.

Lectures de Mécanique. Première Partie. By E. JOUGUET. Paris, Gauthier-Villars, 1908. x + 210 pp.

THIS book is based upon the idea that "it is very instructive for one who wishes to comprehend the nature of the principles and laws of mechanics to follow the history of their development." The subject matter is presented mainly by long quotations from the original authors, interspersed and followed by comments of the writer, who believes it is interesting to study the classic writers on mechanics as one studies the classic writers in literature. The quotations occupy about 163 of the 210 pages in the text. M. Jouguet has in view the physical aspect of the subject and limits himself to the fundamental principles and essential laws. The first part is devoted to the beginnings of mechanics. After a brief introduction on the mechanics of the ancients there are three chapters on statics and four on dynamics.

The chapters on statics are devoted to the lever, the parallelogram of forces, and the principle of virtual work. The conception of each physical law is traced in the problems which have suggested it; the final statement of the law being given in the words of the author who first formulated it definitely. For example the first suggestion of the principle of virtual work is found in Aristotle's treatment of the lever. The evolution of ideas leading to the statement of the principle is then traced by quotations from the writings of Leonardo da Vinci, Cardan, Galileo, and others, its final formulation being given by Descartes.

The first chapter on dynamics contains long quotations from Galileo and Descartes presenting the idea of inertia. The second chapter deals with the phenomena of direct impact from which are deduced the law of action and reaction and the determination of the velocity after collision. The last two chapters treat of the center of oscillation, force and acceleration, and kinetic energy.

Throughout the book attention is called to the different ways of reaching a conclusion, the contrast being particularly striking in the development of the law of inertia, where the method of Descartes is characterized as metaphysical while the inferences of Galileo are drawn from physical observations and experiments. Some of the false conclusions of the early thinkers are presented in connection with the correct results which have led to the statement of modern fundamental principles.

W. R. LONGLEY.

Cours d'Astronomie. Seconde Partie : *Astronomie pratique.* By H. ANDOYER. Paris, A. Hermann & Fils. 1909. 299 pp.

THIS volume completes the course, of which the first part deals with theoretical astronomy. As a text-book for a first course in practical astronomy, the subject matter of the second part is well chosen. Only the common instruments, namely, the theodolite, the equatorial, and the meridian transit are treated in detail, the more special instruments, such as the heliometer and the siderostat, receiving very brief mention. No cuts of astronomical instruments appear in the book and the question of construction is dismissed with the remark that a few hours' acquaintance with the instrument itself is more profitable than a study of the most minute description. The theory of the constants and errors of the instruments and their adjustment is thoroughly worked out. Very naturally the problems which receive the most attention are those connected with the determination of the geographical position of the observer, although a rather brief discussion is given of the observations used in the determination of the fundamental constants, including refraction, aberration, nutation, precession, obliquity of the ecliptic, position of the vernal equinox, and parallax of moon, sun, and stars. In addition to the subject matter of observational astronomy the book contains an introductory chapter on numerical calculation, including the theory of interpolation and the method of least squares. The last chapter is an exposition of Gauss's method for determining the elements of an elliptic or parabolic orbit from three complete observations.

The material in the book is not new, but its careful selection and clear presentation make it valuable to the instructor of beginning classes in practical astronomy.

W. R. LONGLEY.

NOTES.

AT the Colloquium of the AMERICAN MATHEMATICAL SOCIETY, to be held at Princeton University, September 15–17, courses of lectures will be delivered by Professor G. A. BLISS, on "Fundamental existence theorems," and Professor EDWARD KASNER on "Geometric aspects of dynamics." Professor J. H. JEANS having resigned his position at Princeton, the course of lectures announced to be given by him has been cancelled.

THE seventy-ninth annual meeting of the British Association for the Advancement of Science will be held in Winnipeg during the week August 25 to September 1. Sir J. J. THOMSON is president of the association for this meeting and Professor E. RUTHERFORD is president of Section A (Mathematics, Physics and Astronomy). Low rates are quoted by the Canadian railroads and there will be a number of excursions including one to the Pacific coast. Abstracts of papers, which the authors desire to have printed for use at the meetings, should be in the hands of the Secretary of Section A, Professor J. C. FIELDS, University of Toronto, early in June. For information other than that connected with the programme, application should be made to the "Local Secretaries of the British Association," Winnipeg, Manitoba.

AT the meeting of the London mathematical society held on April 22 the following papers were read: By F. TAVANI, "The general principles of the theory of integral equations"; by H. R. HASSÉ, "The equations of electrodynamics and the null influence of the earth's motion on optical and electrical phenomena"; by S. N. WATSON, "The solution of a certain transcendental equation"; by H. BATEMAN, "The physical applications of certain conformal transformations of a space of four dimensions and the representation of a space time point by means of a sphere"; by A. E. WESTERN, "Some criteria for the residues of eighth and other powers"; by W. H. YOUNG, "On the discontinuities of a function of one or more real variables."

KING LEOPOLD, of Belgium, offers a prize of $5,000 to the author of the best work on the subject: "Describe the progress of aerial navigation and the best means to encourage it." Competing memoirs may be written in any of the better known languages of Europe. They should be sent to the minister of sciences and arts at Brussels before March 1, 1911.

THE council of the American federation of teachers of the mathematical and the natural sciences met in Baltimore December 28, 1908. Reports were heard from the local organizations, and a memorial presented to Congress, urging that the board of education be given more extensive power. Professor H. W. TYLER, of the Massachusetts Institute of Technology, was elected president, and Professor C. R. MANN, of the University of Chicago, secretary-treasurer for the present year.

The following advanced courses in mathematics are announced for the year 1909–1910:

HARVARD UNIVERSITY.—Each course is three hours per week.—By Professor W. E. BYERLY: Introduction to modern geometry and modern algebra; Advanced calculus; Trigonometric series (with Professor Peirce).—By Professor W. F. OSGOOD: Theory of functions, I; Linear differential equations.—By Professor C. L. BOUTON: Theory of numbers (first half year); Elementary differential equations (second half year); Geometric transformations.—By Professor J. K. WHITTEMORE: Differential geometry of curves and surfaces (first half year); Properties of polynomials and invariants (second half year); Celestial mechanics (first half year).—By Professor J. L. COOLIDGE: Probability; Algebraic plane curves.—By Professor E. V. HUNTINGTON: Fundamental concepts of mathematics (second half year).—By Dr. H. N. DAVIS: Elements of mechanics.—By Mr. G. C. EVANS: Vector analysis and quaternions. Courses of reading and research are offered by Professors Byerly, Osgood, Bouton, Whittemore, and Coolidge. A seminary in geometry will be conducted by Professors Bouton, Whittemore, and Coolidge.

JOHNS HOPKINS UNIVERSITY. — By Professor F. MORLEY: Higher geometry, two hours; Theory of functions, two hours (first half year); Vector analysis, two hours (second half year). — By Dr. A. COHEN: Differential equations, two hours; Elementary theory of functions, two hours. — By Dr. A. B. COBLE: Theory of groups, two hours; Theory of probabilities, two hours (first half year).

UNIVERSITY OF PENNSYLVANIA (Summer School, July 6 to August 17, 1909). Each course one hour daily. — By Professor I. J. SCHWATT: Infinite series and products.— By Professor G. H. HALLETT: Modern analytic geometry.— By Professor F. H. SAFFORD: Differential equations.— By Dr. O. E. GLENN: Invariant algebra.

PROFESSOR H. E. TIMERDING, of the University of Strassburg, has accepted a professorship of mathematics at the technical school of Braunschweig.

DR. W. VOGT has been appointed docent in mathematics at the technical school at Karlsruhe.

Dr. W. KUTTA, of the technical school at Munich, has accepted an associate professorship of mathematics at the University of Jena.

AT the meeting of the National academy of sciences held at Washington April 22–24 the following mathematicians and physicists were elected to membership: J. S. AMES, of Johns Hopkins University; MAXIME BÔCHER, of Harvard University; OSKAR BOLZA, of the University of Chicago; H. CREW, of Northwestern University.

PROFESSOR W. F. OSGOOD, of Harvard University, has been elected corresponding member of the Mathematical Society of Charkow.

DR. C. L. E. MOORE, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

AT Williams College Dr. F. L. GRIFFIN has been promoted to an assistant professorship of mathematics.

MR. L. L. SILVERMAN, of the University of Missouri, has been appointed instructor in mathematics at Cornell University.

PROFESSOR MAXIME BÔCHER, of Harvard University, has been granted a leave of absence for the academic year 1909–1910; he will spend the year in study in Europe.

PROFESSOR H. v. STAHL, of the University of Tübingen, died April 6, at the age of 65 years.

RECENT second-hand catalogues of mathematical works: A. Hermann & Fils, 6, rue de la Sorbonne, Paris, Bulletin No. 97, about 1,500 titles.—Conrad Skopnik, Prinz Louis Ferdinandstrasse 1, Berlin, N. W., Catalogue No. 43, 731 titles in mathematics, physics, and astronomy.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BÜCHEL. See MITTEILUNGEN.

FINKE (P.). Ueber Scharen von ∞^5 Kurven im gewöhnlichen Raume. (Diss.) Berlin, Frenkel, 1909. 8vo. 36 pp M. 2.00

GROSS (T.). See STURM (C.).

HOFFMANN (F.). Der Satz von Fermat. Strassburg, 1909. 8vo. M. 0.50

HOPPE. See MITTEILUNGEN.

MELLOR (J. W.). Higher mathematics for students of chemistry and physics. London, Longmans, 1909. 8vo. 664 pp. Cloth. 15s.

MITTEILUNGEN der mathematischen Gesellschaft in Hamburg. Vol. IV. 9tes Heft. Redigiert von Hoppe, Büchel und Umlauf. Leipzig, Teubner, 1909. 8vo. Pp. 403–456. M. 1.60

OSGOOD (W. F.). Differential and integral calculus: a first course. Revised edition. New York, Macmillan, 1909. 12mo. 15 + 462 pp. $2 00

STURM (C.). Lehrbuch der Analyse (Cours d'analyse). Uebersetzt von T. Gross. Neue Ausgabe. Berlin, 1909. 8vo. M. 10.00

STURM (R.). Die Lehre von den geometrischen Verwandtschaften. Dritter Band: Die eindeutigen linearen Verwandtschaften zwischen Gebilden dritter Stufe. Leipzig and Berlin, Teubner, 1909. 8vo. 8 + 574 pp. Cloth. M. 20.00

UMLAUF. See MITTEILUNGEN.

VLACHOS (C.). Der Beweis des Fermat'schen Satzes. Berlin, Gottheiner, 1908. 8vo. 7 pp. M. 0.75

VOGT (H.). Eléments de mathématiques supérieures à l'usage des physiciens, chimistes et ingénieurs et des élèves des faculté de sciences. 5e édition. Paris, Vuibert, 1909. 8vo. 8 + 708 pp.

II. ELEMENTARY MATHEMATICS.

Donadt (A.). See Lübsen (H. B.).

Durell (C. V.). A course of plane geometry for advanced students. Part I. New York, Macmillan, 1909. 8vo. 11 + 219 pp. $1.50

Gauss (F. G.). Fünfstellige vollständige logarithmische und trigonometrische Tafeln. Zum Gebrauche für Schule und Praxis bearbeitet. 100ste Auflage. Halle, Strien, 1909. 8vo. 2 + 176 + 35 pp. Cloth. M. 2 50

Hupe (A.). See Müller (H.).

Lesser (O.). See Schwab (K.).

Lormeau (E.) et Merlet (P.). Problèmes de géométrie donnés aux examens du baccalauréat, latin-sciences, sciences-langues, mathématiques A (1904–08), énoncés en la géométrie André et Lormeau. Enoncés et solutions. Paris, André. 8vo. 279 pp. Fr. 4.00

Lübsen (H. B.). Ausführliches Lehrbuch der Elementar-Geometrie. Zum Schul- und Selbstunterricht mit Rücksicht auf die Zwecke des praktischen Lebens in 2 Teilen bearbeitet. 30ste Auflage, neubearbeitet von A. Donadt. Leipzig, Brandstetter, 1909. 8vo.
1. Ebene Geometrie. 8 + 176 pp. M. 2.80
2. Körperliche Geometrie. Mit einer Anleitung zum perspektiven Zeichnen. 2 + 122 pp. M. 2.00

Merlet (P.). See Lormeau (E.).

Metzig (C.). Auflösungen und Lösungsanleitungen zu dem Lehrbuche der Arithmetik und Algebra, verbunden mit einer Aufgabensammlung von mehr als 5800 Aufgaben. Berichtigt nach der 3ten Auflage des Lehrbuches. Breslau, Morgenstern, 1909. 8vo. 56 pp. M. 1.30

Minet (A.) et Patin (L.). Cours pratique d'arithmétique, de système métrique, et de géométrie. Cours moyen. 21e édition, revue, corigée et augmentée de problèmes de récapitulation. Paris, Nathan, 1909. 16mo. 314 pp.

Müller (H.). Die Mathematik auf den Gymnasien und Realschulen. Für den Unterricht dargestellt. Ausgabe B: Für reale Anstalten und Reformschulen, unter Mitwirkung von A. Hupe. (H. Müller's mathematisches Unterrichtswerk.) Abteilung II der Oberstufe: Geometrie der Kegelschnitte. Darstellende Geometrie. 3te Auflage. Leipzig, Teubner, 1909. 8vo. 8 + 178 pp. M. 2.40

Naber (H. A.). Das Theorem des Pythagoras, wiederhergestellt in seiner ursprünglichen Form und betrachtet als Grundlage der ganzen Pythagoräischen Philosophie. Haarlem, 1909. 8vo. M. 7.00

Patin (L.). See Minet (A.).

Pincherle (S.). Algebra elementare. 10a edizione riveduta. Milano, Hoepli, 1909. 16mo. 8 + 210 pp.

Salomon (A.). Nouvelles leçons de géométrie pratique et théorique à l'usage de l'enseignement secondaire des jeunes filles (classes de 3e et 4e années) et de l'enseignement primaire supérieur. 2e édition. Paris, Vuibert, 1909. 16mo. 10 + 302 pp.

Schuster (M.). Geometrische Aufgaben und Lehrbuch der Geometrie: Planimetrie; Stereometrie; ebene und sphärische Trigonometrie. Nach konstruktiv-analytischer Methode bearbeitet. Ausgabe A: Fur Vollanstalten. 1ter Teil: Planimetrie. 3te Auflage. Leipzig, Teubner, 1908. 8vo. 10 + 158 pp. M. 2 20

SCHWAB (K.) und LESSER (O.). Mathematisches Unterrichtswerk zum Gebrauche an höheren Lehranstalten. Im Sinne der Meraner Lehrpläne bearbeitet. Vol. I: Lehr- und Uebungsbuch für den Unterricht in der Arithmetik und Algebra. 1ter Teil: Für die mittleren Klassen samtlicher höheren Lehranstalten. 2te, durchgesehene Auflage. Leipzig, Freytag, 1909. 8vo. 203 pp. M. 2.80

SIMON (P.). Traité de géométrie élémentaire à l'usage des élèves de l'enseignement secondaire (premier et second cycles), suivi de compléments à l'usage des candidats aux écoles du gouvernement. 2e édition. Paris, Belin, 1909. 8vo. 451 pp. Fr. 4.00

SOCCI (A.) e TOLOMEI (G.). Aritmetica generale e algebra: libro di testo per i licei. Vol. II, per la 2a classe. Firenze, Le Monnier, 1909. 8vo. 106 pp. L. 1.25

TOLOMEI (G.). See SOCCI (A.).

III. APPLIED MATHEMATICS.

APPELL (P) et CHAPPUIS (J.). Leçons de mécanique élémentaire à l'usage des élèves des classes de mathématiques A et B, conformément aux programmes de 1905. 1re partie: Notions géométriques; Cinématique. 3e édition, entièrement refondue. Paris, Gauthier-Villars, 1909. 16mo. 9 + 180 pp. Fr. 2.75

BERNIOLLE (P.). Leçons de géométrie descriptive, conformes aux programmes de 1905. Pour les classes de première C et D. 3e édition. Paris, Paulin, 1909. 18mo 194 pp. Fr. 2.50

BOCQUET (J. A.). Corso elementare di meccanica applicata, ad uso delle scuole professionali, industriali, tecniche e degli operai. Traduzione di F. Sinigaglia. 2a edizione, riveduta e corretta sulla 5a edizione francese. Napoli, Pellerano, 1909. 8vo. 6 + 359 pp.

BOULANGER (A.). Hydraulique générale. Vol. I: Principes et problèmes fondamentaux. Paris, Doin, 1909. 18mo. 16 + 382 pp.

CAPART (G.). See VIEWEGER (H.).

CHAPPUIS (J.). See APPELL (P.).

EGERER (H.). Neue Methoden der Berechnung ebener und räumlicher Fachwerke. Berlin, Springer, 1909. 8vo. 7 + 96 pp. M. 2.40

GUAZZUGLI (M. G.). Nozioni fondamentali d'idrologia. Roma, Armani, 1909. 8vo 104 pp.

HAEDER (H.). Konstruieren und Rechnen. Für Studium und Praxis bearbeitet. 4te, erweiterte Auflage. 2 vols. Wiesbaden, Haeder, 1909.
1. Maschinenelemente. 8vo. 16 + 396 pp. Cloth. M. 12.50
2. Beispiele und Tafeln. 11 + 296 pp. Cloth. M. 15.00

JANTZEN (J.). Berechnung und Konstruktion der Einspritz-Kondensatoren und Luftpumpen. Lehrbuch für Studierende des Maschinenbaues und gleichzeitig ein Handbuch für ausübende Techniker und Ingenieure. Hannover, Jänecke, 1909. 8vo. 4 + 113 pp. Cloth. M. 6.00

JOUGUET (E.). Lectures de mécanique. La mécanique enseignée par les auteurs originaux. 2e partie: L'organisation de la mécanique. Paris, Gauthier-Villars, 1909. 8vo. 290 pp. Fr. 10.00

KREUTER (F.). Beitrag zur Berechnung und Ausfuhrung der Staumauern. München, Oldenbourg, 1909. 8vo. 54 pp. Cloth. M. 2.40

KURZWEIL (L.). Die Theorie des Baumeisters; Mathematik und Festigkeitslehre. Für die Baumeisterprüfung und zum Gebrauche für Architekten, Baumeister und Bautechniker in der Praxis. Wien, Graefer, 1909. 8vo. 8 + 172 pp. Cloth. M. 4.80

LANGE (W.). Sammlung von mathematischen Formeln und bautechnischen mathematisch zu lösenden Aufgaben. Ein Hilfsbuch fur Bautechniker und Baugewerkschuler zur Wiederholung und zum Selbstunterricht. 3te Auflage. Bremen, Winter, 1909. 8vo. 107 pp. Cloth. M. 3.50

RICHARD (G.). Mécanique, revue. 31e édition. Paris, Dunod, 1909. 16 mo. 5 + 229 + 61 pp. Fr. 2.50

SINIGAGLIA (F.). See BOCQUET (J. A.).

STEINMETZ (C. P.). Theory and calculations of transient electric phenomena and oscillations. New York, McGraw, 1909. 8vo. 25 + 556 pp. $5.00

SYMPOSIUM on mathematics for engineering students; being the proceedings of the joint sessions of the Chicago section of the American Mathematical Society and Section A, mathematics, and Section D, mechanical science and engineering, of the American Association for the Advancement of Science held at the University of Chicago, Dec. 30–31, 1907. American Mathematical Society, Chicago section, 1909. 4to. 55 pp. Gratis.

TOURNOIS (A.). Précis de mécanique à l'usage des élèves des classes de mathématiques A et B, des candidats au baccalauréat, aux Ecoles de Saint-Cyr, Navale, de l'Institut agronomique (programme de 1905). Paris, Delagrave, 1909. 8vo. 168 pp. Fr. 2.75

VIEWEGER (H.). Recueil de problèmes avec solutions sur l'électricité et ses applications pratiques. Traduction française de G. Capart. Paris, Dunod, 1909. 8vo. 20 + 309 pp. Fr. 9.00

VIGILA. Note di meccanica, 1908–09 [per gli alunni del liceo]. Brescia, Geroldi, 1909. 8vo. 55 pp.

XAVIER (A.). Théorie des approximations numériques et du calcul abrégé. Paris, Gauthier-Villars, 1909. 8vo. 291 pp. Fr. 10.00

TAUTOCHRONES AND BRACHISTOCHRONES.

BY PROFESSOR EDWARD KASNER.

(Read before the American Mathematical Society, February 27, 1909.)

IN the simplest case of a particle acted upon by gravity, the tautochrone problem, solved by Huygens in 1673, and the brachistochrone problem, solved by Jean Bernoulli in 1697, give rise to the same curves, namely, cycloids with horizontal bases and concavity upwards. In the case of a general field of force the two problems lead to distinct systems of curves. Their differential equations, each of the third order, are given in the first section of this note. In § 2 it is shown that the only force besides gravity for which the two systems coincide is the central force varying directly as the distance from the origin. If the force generating the brachistochrones is not required to be the same as that generating the tautochrones, then a third case of duplication is possible (§ 4). Incidentally, a class of forces involving eight parameters and related to infinitesimal collineations presents itself (§ 3); they are the only fields of force for which every straight line of the plane is a tautochrone.

§ 1. *General Equations.*

We consider a particle of unit mass moving in the plane under a force whose rectangular components are $\phi(x, y)$, $\psi(x, y)$. With reference to an arbitrary curve the normal and tangential components are

$$(1) \qquad N = \frac{\psi - y'\phi}{\sqrt{1 + y'^2}}, \quad T = \frac{\phi + y'\psi}{\sqrt{1 + y'^2}}.$$

The condition for a tautochrone is that the motion along the curve be harmonic, that is,

$$(2) \qquad T = \kappa(s - s_0),$$

where κ is a constant* and $s - s_0$ denotes the arc reckoned from a fixed point of the curve, the center of the tautochronous motion. Differentiating twice with respect to s, we have, as the equation of all the tautochrones,

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* For an *actual* tautochrone κ must be negative. The differential equation (3) applies also when κ is positive. Such curves may be termed *virtual* tautochrones. They are the actual tautochrones of the reversed force. A similar remark applies to trajectories, brachistochrones, and catenaries.

$$T_{ss} = 0. \tag{3}$$

The brachistochrones may be found most readily from Euler's theorem on pressure, which gives

$$\frac{v^2}{r} = -N, \tag{4}$$

where v is the speed and r the radius of curvature.

Differentiating both sides with respect to s, and remembering that

$$vv_s = \frac{dv}{dt} = T, \tag{5}$$

we find the equation of the family of brachistochrones to be *

$$2T + rN_s + Nr_s = 0. \tag{6}$$

Equations (3) and (6) when expanded are seen to involve third derivatives of y with respect to x, so that each defines a triply infinite system of curves.

To obtain the results in more explicit form we introduce an auxiliary vector, completely defined by the given field of force, namely, the space derivative of the force (considered as a vector). The rectangular components of the new vector are

$$\phi_s = \frac{\phi_x + y'\phi_y}{\sqrt{1 + y'^2}}, \quad \psi_s = \frac{\psi_x + y'\psi_y}{\sqrt{1 + y'^2}}, \tag{7}$$

and its normal and tangential components are

$$\begin{aligned} \mathfrak{N} &= \frac{\psi_s - y'\phi_s}{\sqrt{1 + y'^2}} = \frac{\psi_x + (\psi_y - \phi_x)y' - \phi_y y'^2}{1 + y'^2}, \\ \mathfrak{T} &= \frac{\phi_s + y'\psi_s}{\sqrt{1 + y'^2}} = \frac{\phi_x + (\phi_y + \psi_x)y' + \psi_y y'^2}{1 + y'^2}. \end{aligned} \tag{8}$$

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* This result, like Euler's theorem, applies only to conservative forces. Apparently the theory of brachistochrones for non-conservative fields has not been investigated. The system in this case consists in fact of ∞^4 instead of merely ∞^3 curves. Since the tautochrone system always contains ∞^3 curves, such forces need not be considered in the problems of § 2 and § 4.

The following noteworthy relations may be derived without difficulty:

$$(9)\qquad N_s = \mathfrak{N} - \frac{T}{r}, \qquad T_s = \mathfrak{T} + \frac{N}{r}.$$

We require finally the space derivatives of $\mathfrak{N}$ and $\mathfrak{T}$, which may be written in the form

$$(10)\qquad \mathfrak{N}_s = \mathfrak{N}_1 + \frac{\mathfrak{N}_2}{r}, \qquad \mathfrak{T}_s = \mathfrak{T}_1 + \frac{\mathfrak{T}_2}{r},$$

where

$$(11)\qquad \begin{aligned} \mathfrak{N}_1 &= \frac{\psi_{xx} + (2\psi_{xy} - \phi_{xx})y' + (\psi_{yy} - 2\phi_{xy})y'^2 - \phi_{yy}y'^3}{(1+y'^2)^{\frac{3}{2}}}, \\ \mathfrak{N}_2 &= \frac{\psi_y - \phi_x - 2(\phi_y + \psi_x)y' + (\phi_x - \psi_y)y'^2}{1+y'^2}, \\ \mathfrak{T}_1 &= \frac{\phi_{xx} + (2\phi_{xy} + \psi_{xx})y' + (\phi_{yy} + 2\psi_{xy})y'^2 + \psi_{yy}y'^3}{(1+y'^2)^{\frac{3}{2}}}, \\ \mathfrak{T}_2 &= \frac{\phi_y + \psi_x + 2(\psi_y - \phi_x)y' - (\phi_y + \psi_x)y'^2}{1+y'^2}. \end{aligned}$$

The functions ϕ, ψ depend only on the position of the particle; the auxiliary functions N, T, $\mathfrak{N}$, $\mathfrak{T}$, $\mathfrak{N}_1$, $\mathfrak{N}_2$, $\mathfrak{T}_1$, $\mathfrak{T}_2$, defined above, depend also upon the direction of motion; finally, N_s, T_s, $\mathfrak{N}_s$, $\mathfrak{T}_s$ depend upon the curvature of the path.

Making use of (9) and (10), we may reduce our equations (3) and (6) to explicit form, and obtain this result:

The general equation of the system of tautochrones is

$$(\text{I})\qquad Nr_s = \mathfrak{T}_1 r^2 + (\mathfrak{T}_2 + \mathfrak{N})r - T,$$

while that of the system of brachistochrones is

$$(\text{II})\qquad Nr_s = -\mathfrak{N}r - T.$$

From these equations we may deduce the general geometric properties of the two systems. We note merely that for tautochrones r_s (the rate of variation of the radius of curvature per unit of arc) is a quadratic function of r, while for brachistochrones it is a linear function of r. The results (I) and (II) are of course differential equations of the third order with respect to y as a function of x. The usual notation may be obtained by substituting

$$(12) \qquad r = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''}, \quad r_s = \frac{3y'y''^2 - (1 + y'^2)y'''}{y''^2},$$

but the above intrinsic forms are more convenient in the following.

We note in passing that for any field of force the equation

$$(13) \qquad N r_s = -\mathfrak{N} r + mT,$$

involving an arbitrary parameter m, includes the following four distinct systems of curves of physical interest: brachistochrones when $m = -1$, trajectories when $m = 3$, catenaries when $m = 2$, and velocity curves when $m = 1$. For all values of m the curves may be obtained by requiring that the pressure shall vary directly as the normal component of the external force. No cases of duplication, beyond those already mentioned, are found by comparing these systems with tautochrones or with one another.

§ 2. *The Duplication Problem for a Single Field.*

We proceed to find the forces for which the brachistochrone system coincides with the tautochrone system. The conditions expressing the equivalence of equations (I) and (II) are

$$(14) \qquad \mathfrak{T}_1 = 0,$$

$$(15) \qquad \mathfrak{T}_2 + 2\mathfrak{N} = 0.$$

Expanding the latter, we find

$$(15') \qquad \phi_y + 3\psi_x = 0, \quad \psi_y - \phi_x = 0, \quad 3\phi_y + \psi_x = 0.$$

The solutions are easily found to be

$$(16) \qquad \phi = cx + a, \quad \psi = cy + b,$$

where a, b, c are arbitrary constants. No new restrictions are imposed by (14). By a simple change of axes and scale the field (16) may be written in one of these two canonical forms,

$$(17) \qquad \phi = 0, \quad \psi = 1,$$

$$(18) \qquad \phi = x, \quad \psi = y.$$

The only types of force for which the tautochrones coincide with the brachistochrones are represented by (17) *and* (18). *In the first type the force is constant in direction and intensity* (*gravity*); *in the second type the force is central and varies directly as the distance from the center* (*elastic law*).

The system of curves in the first type consists of cycloids with horizontal bases; in the second type the curves are hypocycloids and epicycloids, including as limiting cases equiangular spirals, all having the origin as center.

§ 3. *A Class of Forces Related to Collineations.*

We now determine those forces for which the right hand member of the general tautochrone equation (I) is linear in r.* The condition for this is $\mathfrak{T}_1 = 0$, which decomposes on expansion into the set of partial differential equations

$$(19) \qquad \begin{aligned} \phi_{xx} = 0, \quad 2\phi_{xy} + \psi_{xx} = 0, \\ \phi_{yy} + 2\psi_{xy} = 0, \quad \psi_{yy} = 0. \end{aligned}$$

The integration of this set may be carried out without difficulty; but the calculation may be avoided by noticing a simple connection with the equations arising in the determination of infinitesimal collineations. If an infinitesimal transformation, represented symbolically by

$$\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y},$$

is to be projective, then the well known conditions are

$$\begin{aligned} \eta_{xx} = 0, \quad 2\eta_{xy} - \xi_{xx} = 0, \\ \eta_{yy} - 2\xi_{xy} = 0, \quad \xi_{yy} = 0. \end{aligned}$$

This is converted into our set (17) by replacing ξ by ψ and η by $-\phi$. Hence the solutions of (17) are

$$(20) \qquad \begin{aligned} -\phi = b + ex + gy + hxy + ky^2, \\ \psi = a + cx + dy + hx^2 + kxy, \end{aligned}$$

involving eight arbitrary constants, the notation being taken in agreement with Lie-Scheffers, Continuierliche Gruppen, page 24. Our result is then:

---

* The linear form is equivalent to property *I* of a system of dynamical trajectories. See *Trans. Amer. Math. Soc.*, vol. 7 (1906), p. 405.

If a system of tautochrones is to have the property that r_s *(the rate of variation of the radius of curvature per unit of arc) is a linear function of* r *(the radius of curvature), then the field of force must be of the class* (20). *These fields are characterized by the fact that the related infinitesimal transformation*

$$\psi \frac{\partial}{\partial x} - \phi \frac{\partial}{\partial y} \tag{21}$$

is a collineation.

It follows that the lines of force,

$$\frac{dy}{dx} = \frac{\psi}{\phi}, \tag{22}$$

have for orthogonal trajectories a system of anharmonic curves (W curves). Of course this property is not characteristic, since it describes only the direction of the force, not its intensity.

The forces (20) arise most concretely in response to this problem: In what cases will every straight line be a possible tautochrone? *

For a straight line we have $r = \infty$, and $r_s = 0$. If (I) is to be satisfied by these values, then $\mathfrak{T}_1$, the coefficient of r^2, must vanish. The same result is obtained by reducing (I) to the form

$$y''' = \mathfrak{T}_1 + y'' f_1(x, y, y') + y''^2 f_2(x, y, y'), \tag{23}$$

and imposing the condition that $y'' = 0$ shall be a particular integral.

The class of forces (20) *is thus completely characterized by the fact that every straight line of the plane is a possible tautochrone.*

If the force (20) is required to be conservative it is found that the coefficients must satisfy the relations

$$h = 0, \quad k = 0, \quad c + g = 0. \tag{24}$$

The corresponding collineation then becomes an affine transformation leaving areas invariant. The components of the force are of the linear form

$$\phi = -b - ex + cy, \quad \psi = a + cx + dy$$

and the work function is

---

* Mr. Reddick has shown that in space of three dimensions this problem leads to a class of forces involving twenty parameters; the relation to collineations is therefore peculiar to the plane.

$$W = -bx + ay + cxy - \tfrac{1}{2}ex^2 + \tfrac{1}{2}dy^2. \tag{25}$$

The only conservative forces for which every straight line is a tautochrone are those in which the potential is a quadratic function of x, y.

The lines of force are then the orthogonals of a system of homothetic conics. Such forces are of importance in connection, for example, with the general theory of motion about a position of equilibrium.

§ 4. *The Duplication Problem for Two Fields.*

When will the system of ∞^3 tautochrones connected with one field of force coincide with the system of ∞^3 brachistochrones connected with a second field? We denote the components of the first force by ϕ, ψ, and those of the second force by ϕ^*, ψ^*, so that the differential equations of the two systems considered are

$$N r_s = \mathfrak{T}_1 r + (\mathfrak{T}_2 + \mathfrak{N}) r - T, \tag{26}$$

$$N^* r_s = -\mathfrak{N}^* r - T^*. \tag{27}$$

We observe that a necessary condition for equivalence is $\mathfrak{T}_1 = 0$; hence the first force must be of the type (20). The additional conditions to be satisfied are

$$\frac{T^*}{N^*} = \frac{T}{N}, \quad \frac{\mathfrak{T}_2 + \mathfrak{N}}{N} = -\frac{\mathfrak{N}^*}{N^*}. \tag{28), (29}$$

From the first of these, we find that the ratio of ψ^* to ϕ^* must be equal to the ratio of ψ to ϕ. We may therefore write

$$\phi^* = \rho\phi, \quad \psi^* = \rho\psi, \tag{30}$$

where ρ is an unknown function of x, y. A short calculation shows that, in consequence of (30),

$$\frac{\mathfrak{N}^*}{N^*} = \frac{\mathfrak{N}}{N} + \frac{\sigma_x + y'\sigma_y}{\sqrt{1 + y'^2}} \qquad (\sigma = \log \rho). \tag{31}$$

Equation (29) may then be written in the form

$$\mathfrak{T}_2 + 2\mathfrak{N} + N\sigma_s = 0. \tag{32}$$

Expanding this in powers of y', making use of the known form (20) of ϕ and ψ, and equating coefficients to zero, we find

$$
\begin{aligned}
& 3c - g + 5hx + ky + \psi\sigma_x = 0, \\
(32') \qquad & 4(d + e + kx + hy) + \psi\sigma_y - \phi\sigma_x = 0, \\
& 3g - c + hx + 5ky - \phi\sigma_y = 0.
\end{aligned}
$$

The unknowns are now the eight coefficients involved in (20) and the function σ. Elimination of σ_x and σ_y gives

$$
(33) \qquad \begin{aligned}
\phi^2(3g - c + hx + 5ky) + 4\phi\psi(d + e + kx + hy) \\
+ \psi^2(3c - g + 5hx + ky) = 0.
\end{aligned}
$$

By means of the coefficients of the highest powers here involved, namely, x^5 and y^5, we find that h and k must vanish. The last equation thus simplifies to

$$
(33') \qquad (3g - c)\phi^2 + 4(d + e)\phi\psi + (3c - g)\psi^2 = 0,
$$

where

$$
(34) \qquad \phi = -b - ex - gy, \quad \psi = a + cx + dy.
$$

We now divide the discussion into two cases:

Case 1°. If the coefficients $3g - c$, $4(d + e)$, $3c - g$ appearing in (33′) all vanish, then $c = 0$, $g = 0$, $d = -e$, and we obtain the form (16) of § 2. In fact, under these conditions, equations (32′) show that both σ_x and σ_y will vanish, that is, σ and hence ρ will be a constant, that is, the two fields will coincide.

Case 2°. Under the contrary assumption, we see from (33′) that the ratio of ψ to ϕ must be constant. This means that the force has a constant direction, which we may without loss of generality assume to be the direction of the y-axis. The component ϕ will then vanish, so that, from (34),

$$
b = e = g = 0.
$$

Equation (33′) then gives $3g - c = 0$, so that c also must vanish. Our force thus becomes

$$
\phi = 0, \quad \psi = a + dy.
$$

If d vanishes, this becomes the type (17). Otherwise, it may, by a change of axes and scale, be reduced to the form

$$
(35) \qquad \phi = 0, \quad \psi = y.
$$

Substituting these values in (32′) we find that the equations for the unknown function σ are consistent and give

$$
\sigma = -4 \log y,
$$

so that ρ, the factor for converting the first field into the second field, is y^{-4}. Hence the second field is

$$(36) \qquad \phi^* = 0, \quad \psi^* = y^{-3}.$$

In (35) the force varies directly as the distance from the x axis, while in (36) the force varies as the inverse cube of that distance.

The only cases in which a system of tautochrones is also a system of brachistochrones are these three:

1°. *The tautochrones and the brachistochrones of the uniform field* $\phi = 0$, $\psi = 1$ *coincide.*

2°. *The same is true of the elastic field* $\phi = x$, $\psi = y$.

3°. *The tautochrones of the field* $\phi = 0$, $\psi = y$ *coincide with the brachistochrones of the field* $\phi^* = 0$, $\psi^* = y^{-3}$.

All these fields are conservative.

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DEGENERATE PENCILS OF QUADRICS CONNECTED WITH $\Gamma^{n+2}_{n+4,n}$ CONFIGURATIONS.

BY DR. W. B. CARVER.

(Read before the American Mathematical Society, December 31, 1908.)

IN a previous paper[*] the author discussed a certain pencil of quadric spreads associated with the configuration $\Gamma^{n+2}_{n+4,n}$ in space of n dimensions. The $\Gamma^{n+2}_{n+4,n}$ contains $n + 4$ configurations $\Gamma^{n+1}_{n+3,n}$, and with each of these is associated a quadric spread with respect to which its points and S_{n-1}'s are poles and polars. In the case of a *proper* $\Gamma^{n+1}_{n+3,n}$, *i. e.*, one whose points, lines, planes, $\cdots$, and S_{n-1}'s are all distinct, it is evident that the associated quadric spread cannot degenerate into a cone.[†] Hence, for a proper $\Gamma^{n+2}_{n+4,n}$, the individual spreads of the associated pencil cannot be degenerate; but the question naturally arises as to whether the pencil itself, or the quartic $(n - 2)$-way spread through which the quadrics all pass, may be degenerate. It is the object of this paper to answer this question for the cases $n = 1$, 2, and 3.

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[*] "The quadric spreads connected with the configuration $\Gamma^{n+2}_{n+4,n}$." *Amer. Jour. of Mathematics*, vol. 31, pp. 1–17 (January, 1909).

[†] That is, if the quadric spread be represented by a quadratic equation in $n + 1$ homogeneous variables, the discriminant of this equation must not vanish.

We will consider the three-dimensional case first, as affording the best illustration of the method of treatment. We have a pencil of seven quadric surfaces associated with a $\Gamma^5_{7,3}$ configuration. It was shown in the author's previous paper that the pencil of quadrics could be first chosen arbitrarily, and that the configuration was then formally determined. The analytic procedure there described* may be used here. We may choose a pencil of quadric surfaces of any one of the thirteen well-known types,† and see whether the configuration then determined is or is not degenerate.

For example, consider a pencil of quadrics for which the common quartic curve consists of a conic and two lines not intersecting on the conic (type 8 in the Clebsch-Lindemann classification). Seven surfaces of such a pencil may be represented with perfect generality by the equations

$$x_2x_3 + x_3x_4 + x_4x_2 + \lambda_i x_1x_4 = 0 \quad (i = 1, 2, \cdots, 7).$$

Let the point 123 of the configuration be a_1, a_2, a_3. Then the plane 14567 will be

$$\lambda_1a_4x_1 + (a_3 + a_4)x_2 + (a_2 + a_4)x_3 + (\lambda_1a_1 + a_2 + a_3)x_4 = 0;$$

the point 234 will be

$$\lambda_1\lambda_4a_1 + 2(\lambda_1 - \lambda_4)a_4,\ \lambda_4^2a_2 + \lambda_4(\lambda_4 - \lambda_1)a_4,$$
$$\lambda_4^2a_3 + \lambda_4(\lambda_4 - \lambda_1)a_4,\ \lambda_1\lambda_4a_4;$$

the point 345 will be

$$\lambda_2\lambda_5\lambda_4\lambda_1a_1 + 2\,[\lambda_2\lambda_5(\lambda_1 - \lambda_4) + \lambda_1\lambda_4(\lambda_2 - \lambda_5)]a_4,$$
$$\lambda_4^2\lambda_5^2a_2 + \lambda_4\lambda_5(\lambda_4\lambda_5 - \lambda_1\lambda_2)a_4,$$
$$\lambda_4^2\lambda_5^2a_3 + \lambda_4\lambda_5(\lambda_4\lambda_5 - \lambda_1\lambda_2)a_4,$$
$$\lambda_2\lambda_5\lambda_1\lambda_4a_4;$$

and the plane 12367 will be

$$\Sigma_2S_2a_4x_1 + \Sigma_2^2(a_3 + a_4)x_2 + \Sigma_2^2(a_2 + a_4)x_3$$
$$+ [\Sigma_2S_1a_1 + \Sigma_2^2(a_2 + a_3) + 2(\Sigma_2^2 + S_3\Sigma_1 - S_2\Sigma_2)a_4]x_4 = 0,$$

where the Σ's and S's are the ordinary symmetric functions respectively of λ_4, λ_5 and λ_1, λ_2, λ_3. Since the point 123 must lie in the plane 12367, we have the relation

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* Loc. cit., p. 5.

† Clebsch-Lindemann, Vorlesungen über Geometrie, vol. II, p. 215.

$$\Sigma_2 S_3 a_1 a_4 + \Sigma_2^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1 S_3 - \Sigma_2 S_2) a_4^2 = 0.$$

Similarly, the coordinates of the point 123 must satisfy the conditions

$$\Sigma_2' S_3 a_1 a_4 + \Sigma_2'^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1' S_3 - \Sigma_2' S_2) a_4^2 = 0$$

and

$$\Sigma_2'' S_3 a_1 a_4 + \Sigma_2''^2(a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4) + (\Sigma_1'' S_3 - \Sigma_2'' S_2) a_4^2 = 0,$$

where the Σ''s and Σ'''s are symmetric functions respectively of λ_5, λ_6 and λ_6, λ_7.

Each of the 35 points of our configuration must satisfy such a set of three equations. These equations are linear and homogeneous in the quantities $a_1 a_4$, $a_4^2 + a_2 a_3 + a_2 a_4 + a_3 a_4$, and a_4^2. The determinant of the coefficients cannot vanish; for it reduces to

$$\lambda_1^2 \lambda_2^2 \lambda_3^2 \lambda_5^2 \lambda_6^2 (\lambda_6 - \lambda_4)(\lambda_4 - \lambda_7)(\lambda_7 - \lambda_5)$$

and no λ can be zero (for this would give a cone) and no two λ's can be equal. Hence the conditions could only be satisfied if

$$a_1 a_4 = 0, \quad a_4^2 + a_2 a_3 + a_3 a_4 + a_2 a_4 = 0, \quad a_4^2 = 0.$$

But this would make the point 123 lie on one of the two lines which make up part of the quartic curve, and *the same would be true for all the points of the configuration.* Hence we cannot have a proper configuration in this case.

By a similar treatment in the other cases, we obtain the following results:

Proper configurations exist for the types of pencils 1–5, and do not exist for types 6–13. For the general case, type 1, there are eight configurations for any given pencil of seven quadrics;* but for types 2–5 there are fewer configurations for a given pencil, because some of the solutions are absorbed by the special points or lines.

We find a more surprising situation, perhaps, in the plane case. Here we have five types of pencils of conics:

1. Four distinct points of intersection;
2. Two distinct points of intersection, and contact of first order at a third point;
3. Contact of first order at two distinct points;

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* Author's paper, loc. cit., p. 6.

4. One point of intersection, and contact of second order at a second point;

5. Contact of third order.

Proper configurations exist for types 1 and 4, and do not exist for types 2, 3, and 5. For a given pencil of six conics there are four configurations for type 1, but only one for type 4.

For the one-dimensional case, we have the configuration $\Gamma^3_{5,1}$ on a line, and the pencil of quadrics is simply five pairs of points in involution.* We have two types, viz., the fixed points of the involution are distinct or they are coincident. Proper configurations exist for both cases. Given the five point pairs, there are two configurations determined in the first case, but only one in the second. This one-dimensional case is readily seen if the whole figure is projected upon a conic.

ON THE USE OF n-FOLD RIEMANN SPACES IN APPLIED MATHEMATICS.

BY PROFESSOR JAMES McMAHON.

THE object of this article is to show that the conception of a Riemann surface has important physical bearings, and to indicate in a general way what kind of physical problems have been solved or may be solvable by the use of such n-fold surfaces or analogous manifold regions in three dimensions. The most recent work in this line constitutes the highest point yet reached in the application of modern function theory to physical problems. It is very noteworthy that a theory which was developed by following out purely intellectual relations, without any reference to the world of sense, should afterwards find unexpected applications and correspondences in the physical universe.

The conceptions of multiform functions, and of multiple spaces in which such functions are made uniform, furnish elegant solutions of some important problems in the theories of potential, electricity, light, sound, heat, and fluid motion. To give greater clearness to what follows, it may be well to take a simple illustration of a three-valued potential function in two dimensions, and show how to make it one-valued on a three-fold Riemann surface. Let (ρ, θ) be the polar coordinates of a point in a plane, and consider the function $\rho^{\frac{1}{3}} \cos \frac{1}{3}\theta$, which satisfies Laplace's equation, and has in general three real and distinct values at the point (ρ, θ), the second and third values being obtained by changing θ into $\theta + 2\pi$ and $\theta + 4\pi$ in succession. On making successive circuits around the origin, the first value changes continuously into the second, the second into the third, and the third into the first. It will be seen that a pair of values are equal when $\theta = 2\pi$, another pair when $\theta = 4\pi$, and the other pair when $\theta = 6\pi$. Now let us imagine the plane covered over by two other plane sheets in such a way that a revolving radius vector shall trace out the first sheet from $\theta = 0$ to $\theta = 2\pi$, and then by continuous motion trace out the second sheet from $\theta = 2\pi$ to $\theta = 4\pi$, and the third sheet from $\theta = 4\pi$ to $\theta = 6\pi$, passing then into the first sheet again, and so on. The three pairs of coordinates (ρ, θ), $(\rho, \theta + 2\pi)$, $(\rho, \theta + 4\pi)$, which before represented coincident points, are now regarded as representing three underlying points in separate sheets, and these points are mutually accessible only by a circuit around the origin (or winding point). The positive half of the initial line, being the threshold for passage from one sheet to another, is called the common branch line. The given function is now single-valued and continuous over the whole three-sheeted Riemann surface.

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* Author's paper, loc. cit., p. 5, footnote.

In an analogous way, the same function in cylindrical coordinates could be uniformized by taking the axis of z as a winding line, and the initial half plane as a branch membrane, giving rise to a three-fold Riemann space, which fills the whole of our space three times.

The first concrete example that I can find of a multiform potential in three dimensions is contained in a letter from Professor P. Appell to Professor F. Klein, published in the *Mathematische Annalen*, 1887, a few months before Klein delivered his lectures on potential theory (summer 1888). Appell cites a two-valued function that satisfies Laplace's three-dimensional equation, and has a certain circle for a singular line, such that the two values of the function interchange when a circuit is made around this winding line. He says nothing, however, about uniformizing the function by introducing another copy (or fold) of three-dimensional space, and using the singular (or winding line) as the boundary of a branch membrane (or door) from one fold into the other. In 1891 Dr. Pockels, in his treatise on the solutions of the "wave potential" equation for various boundary conditions, pointed out the desirability, from both a mathematical and physical standpoint, of studying the multiform solutions of this equation, and the mode of uniformizing them on Riemann surfaces, or analogous three-dimensional spaces. He suggested that the Green functions for certain boundaries in such spaces might correspond to some unsolved physical problems. It is not unlikely that this idea may have been derived from Klein's lectures on potential theory (1888), of which Pockels made free use, but which I have not yet been able to consult directly, as they have been preserved only in manuscript.

Apparently the first application of the notion of a multiple space to a concrete physical problem was made by Professor A. Sommerfeld in articles in the *Mathematische Annalen*, 1894 and 1896, on heat conduction and diffraction, and in another article in the following year in the *Proceedings of the London Mathematical Society*, on "Multiform potentials in space." In these articles Sommerfeld shows among other things that certain boundary problems, in which the Kelvin image method fails to furnish the true Green function, on account of the existence of more than one pole within the given boundary, can be solved by the conception of the Riemann space such that each fold shall contain only one of the poles (or images). Moreover, he proves, by function theory methods, an important fundamental principle, which has been called Sommerfeld's theorem, namely, that Green's general theorem is applicable without modification to functions that have been uniformized in a Riemann space, provided the winding lines are excluded from the region of integration by means of thin tubes. From this he easily proves the uniqueness of the solution of Dirichlet's problem for any assigned boundary in the multiple space.

Sommerfeld first applies his generalized image method to find Green's function for a wedge-shaped region bounded by two infinite half planes and the infinite sphere, the angle of the planes being $n\pi/m$. Starting with the assigned pole, and reflecting successively with regard to the two planes until the series of images closes up, we find that there are n poles within the given boundary; hence we use an n-fold Riemann space having a pole in each fold, the straight edge being the winding line. The appropriate coordinate system is cylindrical, having the winding line as axis. By introducing into the elementary Green function a complex parameter, and then performing a contour integration, using Cauchy's method of residues, a generalized function is obtained for each of the poles, and these are combined in such a way as to realize the conditions for the required Green's function in the assigned physical region. By a similar method the closely related problem of the diffraction of waves of light or electricity in passing through a wedge shaped medium has been solved by H. S. Carslaw (*Philosophical Magazine*, 1903).

The case in which $n = 2$ and $m = 1$ solves various potential and diffraction problems for the infinite half plane. In particular, the cases of waves issuing from a line source or a point source against the half-plane have been fully worked out by Sommerfeld, Carslaw and others. If the screen extended to infinity both ways, the usual method of images could easily be applied, as the image of the source would not be situated in the given region, and the action of the screen could be perfectly imitated by a combination of the source and its image, the screen being removed. As this is evidently not feasible in the case of the half-plane, we have to place the image in a second fold of a Riemann space, and find the proper Green function as already indicated. It is to be observed that the effects of the two sources have to be combined in an appropriate way according as the screen is perfectly reflecting or perfectly absorbing, so as to fulfill the proper conditions at the surface of the screen, which is then thought of as replaced by a branch membrane (or open door) between the two folds of the Riemann space, one of the folds being the ordinary physical space itself, and the other a copy coincident with it, but supposed capable of sustaining a wave motion of its own independent of the motion at underlying points in the other fold, and only influenced by the energy which enters and leaves by the open door.

In this connection it may be of interest to mention a criticism made by Professor K. Schwarzschild (*Mathematische Annalen*, 1901). Alluding to the case in which the screen is perfectly absorbing he remarks that while the second fold of the Riemann space does seem at first sight to carry away the motion so as to realize the proper conditions at the absorbing screen, yet there is nothing to prevent the latter fold from delivering some of this motion back into the first fold (or natural space) after another circuit around the winding line, thus disturbing the adjustment. Sommerfeld skillfully overcomes this objection (*Zeitschrift*, 1901, page 36) by substituting for the two-fold space an n-fold one, and after forming the proper function, letting n become infinite; the limit so obtained must give a correct result, as it would take an infinite time for the energy to wind back into the first fold. This is equivalent to its passing out of the universe, and thus the perfectly absorbing screen is exactly imitated. The case of the perfectly reflecting screen is not open to any similar objection. Moreover, no objection can be raised against the general method on account of any supposed physical difficulty in the coexistence of different motions in the different folds, for, even if that be conceded, there remains a firm analytical ground for the method, in the generalized Green theorem and Dirichlet principle; and in any case it is not difficult to verify that the function finally obtained satisfies all the prescribed conditions in the assigned physical space.

In the article on multiform potentials already quoted, Sommerfeld next considered the more difficult case of an infinite plane out of which is cut an infinite strip bounded by two parallel lines. The suitable Riemann space has two winding lines, and the appropriate coordinate system is the bipolar system used so skillfully by Maxwell. In a later article (*Zeitschrift*, 1901, page 71) on "the bending of Roentgen rays," Sommerfeld points out that the function found in his previous article does not satisfy all the conditions, and that he has not yet found the true Green function for the region in question, although it is certain that one exists. It may be of interest to state, however, that this "problem of the slit" has been solved by an ingenious method of successive approximation by Schwarzschild in the paper already quoted (*Mathematische Annalen*, 1901). He applies Green's function for the infinite half-plane to each of the two given half-planes, in endless succession, the boundary conditions for each application being selected so as to neutralize the error left by the preceding one; this in turn disturbs the adjustment on the other half-plane, but it is proved that the successive errors diminish rapidly, and that the series is convergent.

The article contributed by Sommerfeld to the London Mathematical Society gave rise to a series of further developments along similar lines by British mathematicians and physicists. The following important papers may be mentioned:

"On multiform solutions of the differential equations of physical mathematics" (*Proceedings of the London Mathematical Society*, volume 30 (1898)) by H. S. Carslaw.

"On electric current sheets" (*Proceedings of the London Mathematical Society*, volume 31 (1899)) by J. H. Jeans.

"On the determination of Green's function by means of cylindrical and spherical harmonics" (*Proceedings of the Edinburgh Mathematical Society*, 1900) by J. Dougal.

"The Green function for a circular disk with applications to electrostatics" (*Cambridge Philosophical Transactions*, volume 18 (1900)) by E. W. Hobson.

"The use of contour integration in the problem of diffraction by a wedge of any angle" (*Philosophical Magazine*, 1903) by H. S. Carslaw.

"On many-valued Newtonian potentials" (*Proceedings of the London Mathematical Society*, 1904) by A. C. Dixon.

"On Sommerfeld's diffraction problem" (*Proceedings of the London Mathematical Society*, 1907) by Horace Lamb.

The curvilinear system used by Hobson in dealing with the circular disk is especially noteworthy. It is the peripolar system introduced by C. Neumann in treating of anchor ring problems. The three families of orthogonal surfaces which correspond to these coordinates consist (1) of a system of spherical bowls with the boundary of the disk as a common rim, and including the disk itself as a limiting member, (2) a system of anchor rings with the rim as a limiting member, and (3) a system of planes passing through the axis of the disk. The appropriate Riemann surface is two-sheeted, having the rim for the winding line and the disk for a branch membrane.

The next problem to be attacked by this method would perhaps be the diffraction of plane waves by an infinite plane screen perforated by a circular aperture; and the same coordinate system would naturally be tried. The elliptic aperture would be much more difficult. Then one might try a case of the still unsolved problem of discontinuous fluid motion in three dimensions, such as the following: A large mass of water (or other incompressible fluid) is moving with uniform velocity in one direction, and is partly obstructed by a fixed circular disk placed perpendicularly to the line of motion; to find the system of stream lines, especially those which pass around the edge and form a surface of discontinuity bounding the region of still water.

A similar problem may also be stated for a circular aperture in a fixed plane; and in both problems water may be replaced by air. It would seem that the idea of a Riemann surface (or space) is well fitted to bridge over a discontinuity, by the introduction of an extra copy of space in which the discontinuity is filled up in some suitable way.

The success, however partial, which has attended the quest for mathematical functions that express more and more complicated physical relationships, tends to strengthen our faith in some deep-lying correspondence between the world of nature and the world of the pure intellect. It remains for mathematicians and physicists to discover new winding lines and open doors.

MATHEMATICAL APPOINTMENTS IN COLLEGES AND UNIVERSITIES.

BY PROFESSOR E. J. WILCZYNSKI.

AT the meeting of the Chicago Section held January 2, 1909, a committee was appointed for the purpose of investigating the possibility of improving the character of mathematical appointments in our colleges and universities. I submitted for the consideration of the members of the committee an adaption of the Italian system which appeared to me to be applicable to American conditions. My views were not shared by the other members of the committee, and our official report has been incorporated in the secretary's report of the April meeting of the Chicago Section.

The suggestions which I offered to the committee may be arranged under three heads.

First. Publication of vacancies.—There should be established in the BULLETIN a special department devoted to the announcement of vacancies. The administrative heads of all colleges and universities should be informed of the existence of this department and should be requested to avail themselves of it. The announcements should be as explicit as possible on the items of title, salary, grade and amount of work, probable date of appointment, address to which applications should be sent, date after which no applications will be considered, whether a specialist is desired or preferred, and if so in what special subject. If any of these items are subject to any uncertainty—if, for instance, as is often the case, an institution is not willing to announce a definite sum as salary because it would be willing to pay widely different salaries according to the exigencies of the case— the vacancy and its general character should nevertheless be announced.

Second. The filling of major positions. — A major position shall be defined as one to which is attached a salary of $2,000 a year or more. It seems to me that the following plan might serve as a basis for discussion: If an institution desires the services of the Society in making a major mathematical appointment, it should send a request to the secretary describing as closely as possible the nature of the position to be filled and the kind of person desired. The secretary should acknowledge the request and receive assurance from the institution that it is willing to bear the expenses of the nominating committee, which expenses would in most cases be very small. Having received this assurance, the secretary should inform every member of the Council of the Society, requesting him to make out and send to the secretary before a specified date a list of six names (members or non-members of the Society) in order of preference, these to stand as his nominations for the nominating committee. Upon the specified day the secretary should tabulate the votes. The person receiving the highest number of votes should be appointed as chairman, and the two following him in the number of votes should be named as members of the nominating committee. It should be provided, however, that no two men employed in the same institution should be members of the committee. All ties, whether for chairmanship or membership, should be decided by the secretary. If two or more men from the same institution are voted into the same committee, the secretary should declare elected that one with the larger number of votes, or in case of a tie cast his own vote so as to decide for one of them, and then proceed to complete the committee from those next lower on the list.

If any person so elected to membership in a nominating committee is unable or unwilling to serve, the committee should be completed by election of the person next below him in the total number of votes. Acceptance of membership on the committee should imply renunciation on the part of the person accepting of any intention on his part of being himself a candidate for the position under consideration.

As soon as the committee has been finally constituted, the fact should be announced and the names of the members published in the BULLETIN. This should be coupled with a request to all applicants to forward their applications, a list of their publications and other documents to the chairman, and as far as possible also to the other members of the committee.

It should be the duty of the committee of three elected in this way to consider and weigh carefully the claims of all applicants. But beyond this, it should in all cases survey the entire field and select three or more men who, according to its opinion, would be the best available persons for filling the position under consideration. It should arrange these men in order of preference, giving a brief discussion of the reasons for its decision. The possibility of bracketing two or three men of apparently equal merit should not be excluded however. The report of the committee should then be forwarded to the institution concerned, for any action which it may choose to take. The list of names recommended by the committee should also be published in the BULLETIN in alphabetic order.

Most of the work of the committee could be done by correspondence. In many cases, however, it would be necessary, and it would always be desirable, for the committee to have an opportunity for oral discussion. It is not likely that an institution willing to take advantage of this plan would object to the expenses arising from this. I merely suggest, however, the possibility that a committee member living at a great distance might, with the approval of the President of the Society, appoint and instruct a substitute to take his place in the oral discussion. It would not be impossible to divide the country into regions and insist upon all members of a nominating committee being residents of one such region. But this plan would give rise to some very serious difficulties and grave objections.

Third. The filling of minor positions. — A minor position is defined as one worth less than $2,000 a year. I can see little prospect of success in any attempt on the part of the Society to assist in making such appointments, on account of the large number of such positions, and the great amount of work and expense that would be involved. The suggestion which has been made of requiring a small fee of applicants and of employing a clerk to look after the matter would defeat our main object of giving *expert* advice on such appointments. We might perhaps think of a standing committee of the Council appointed for a year, to meet upon the days assigned to meetings of the Society, whose duty it would be to make recommendations of this character, if the material in each case were properly collected and sifted beforehand by an intelligent clerical assistant. It seems quite likely, however, that the careful filling of major positions would do away with any great necessity of reform in the methods of filling minor positions.

PICARD'S ALGEBRAIC FUNCTIONS OF TWO VARIABLES.

Théorie des Fonctions algébriques de deux Variables indépendantes. Par ÉMILE PICARD, Membre de l'Institute, Professeur à l'Université de Paris, et GEORGES SIMART, Capitaine de Frégate, Répétiteur à l'Ecole Polytechnique. Tome II. Paris, Gauthier-Villars, 1906. 8vo, v + 528 pp.

As the general scope of this great work has been explained and discussed in a very full review of volume I,* it only remains to give an idea of the large amount of additional material contained in the second volume, which is more than twice the size of the first.

Three principal objects are aimed at. The first is to present a systematic account of the theory of algebraic surfaces as developed by the Italian mathematicians. This theory deals chiefly with systems of curves on any given surface and certain numerical invariants connected with them. The other objects of the book have in view the transcendental side of the theory and deal with two of its main problems, namely, (1) the theory of the double integrals of the second kind, and (2) the integrals of total differentials of the third kind.

The work cannot be regarded as a treatise, since the subject matter with which it deals is still in a formative state and not yet sufficiently matured to admit of such a treatment. Its leading aim is rather to present in an accessible and connected form the numerous investigations of Professor Picard in this field. It accordingly consists in large part of reprints of memoirs and notes previously published in various journals together with such changes and additions as are needed to unite them into a systematic whole. A number of other notes relating to the subject, but not bearing directly on the three problems just mentioned, are collected together in an appendix of twenty pages. An additional note of about forty pages entitled "Sur quelques résultats nouveaux dans la théorie des surfaces algébriques" is contributed by Professors Castelnuovo and Enriques. Its purpose is sufficiently indicated by the title.

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* By Mr. A. Berry in this BULLETIN, 2d series, vol. 5 (June, 1899), pp. 438–451.

The work under review has the peculiar interest and fascination which attaches to a new and broad theory in process of development. The matter is presented with that clearness, precision, and elegance which characterize all the productions of the distinguished author. The reader finds the subject growing under his very eyes. This progression is strikingly evident from the fact that the present volume has appeared in three parts in 1900, 1904, and 1906, the delay in issuing the work having been caused by the need of additional investigations for the purpose of filling important gaps in the theory. So that questions are constantly springing up the solutions of which are reached in subsequent chapters of the book.

The first chapter treats of Noether's theorem relative to the curves and surfaces passing through the points of intersection of two given ones. For the case of three variables the proof here formulated is quite different from that given by Noether. One feature of this proof is, in a variety of forms, characteristic of the methods of the book as a whole. It consists in giving a constant value to one of the variables, then applying results which are known for the case of two variables.

The second chapter is a presentation of the familiar theory of point groups on an algebraic curve. This, together with Chapter III on linear systems of curves in a plane, is preparatory to a similar study of surfaces.

In Chapter IV we begin the first problem of generalization by a study of linear systems of surfaces of a given order. Such a system is defined by the behavior of the individual surfaces in certain base lines and isolated base points. The system is complete or incomplete according as all or only part of the surfaces of the given degree which behave in the prescribed manner are included.

Linear systems of surfaces which are adjoint or subadjoint to a given surface f play an especially important role in the whole work. A surface ϕ is said to be subadjoint to f if any plane section of ϕ is adjoint to the corresponding plane section of f. This definition (as well as that of adjoint), which is quite different from that given by Enriques, lends itself readily to the transcendental theory which is the main object of the book. This may readily be seen from the fact that if ϕ is a subadjoint to f, then the double integral

$$(1) \qquad \int\int \frac{\phi dx dy}{f'_z}$$

is finite at all finite points of f excepting a certain limited number of singular points of the surface.

Adjoint surfaces $Q = 0$ of order $m - 4$ are such that when the polynomial Q replaces ϕ in (1), the integral is of the first species, that is, it is everywhere finite. This is called the canonical system. Each surface of the system behaves like a subadjoint of order $m - 4$ along the base lines, and has in addition a certain behavior at the singular points of f determined by the condition that the double integral shall be finite. The number of linearly independent adjoints Q is an invariant and is denoted by p_g. Adjoint surfaces $q = 0$ of order $m - 4 + r$ are defined so that the integral $\int\int q dx dy / f'_z$ is finite at every finite point of f, and hence they behave in the multiple lines and points of f in just the same way as adjoints of order $m - 4$.

Suppose that the surfaces adjoint to f and of order h greater than $m - 4$ are cut by an arbitrary plane. The system of plane curves of order h thus obtained may not be a complete system. Let ω_h denote its default, this being the difference between the number of linear parameters of a complete system and the number of parameters in the specified system. This default becomes zero when h exceeds a certain finite integer $l - 1$. A new invariant p_n for the surface f, called the numeric genus, is introduced and is found to be connected with p_g by the relation

$$p_g - p_n = \sum_{h=m-3}^{l-1} \omega_h.$$

A remarkable theorem, due to Castelnuovo, states that when ω_{m-3} is zero all the other defaults are zero at the same time and in that case the numeric genus is equal to the geometric genus.

Chapter V takes up the linear systems of curves on a given surface $f = 0$. These are formed by the intersection of f with a linear system of surfaces. Such a system may have base points and fundamental curves. The latter occur when the general curve of the system breaks up into a variable part, and a fixed part to which this name is given. A fundamental method in the study of f is to transform it birationally into a surface f' in hyperspace so that the curves of a given system $|C|$ on f become the curves of section of f' by hyperplanes. Of special interest are those linear systems of curves which have no base points or fundamental curves. These always exist. When such a system is used in passing from f to f' we may obtain, by successive and properly chosen projections of f' into spaces of lower dimensions, a surface F in three-dimensional space which has only ordinary singularities (double curves having triple points) and which has a $(1, 1)$ correspondence with f.

The important place occupied by the adjoint and subadjoint surfaces becomes more manifest from the theorem that any linear system of curves can always be obtained as the intersections of f with a linear system of adjoint or subadjoint surfaces of sufficiently high order, subject, if necessary, to pass through certain bases.

Another theorem of fundamental importance is this: If an irreducible linear system of curves on f is not complete, there exists one, and but one, complete system which contains it totally. The proof given by MM. Picard and Simart is strikingly brief and elegant. It is almost intuitive in character and is a great simplification of the quite complicated proof given by Enriques. This is one of various instances in which the theory has been simplified and brought out in greater clearness in the present work.

The theory of point groups, which plays such an important part in the case of plane curves, is extended to curves on a surface f. Any curve of a given linear system is intersected by the other curves of the same system in a series of point groups called the characteristic series. The canonical series is formed by the groups of $2\pi - 2$ variable points, π being the genus of the general curve C of the system. This series is determined by means of the following theorem which holds for $p_g > 0$: An arbitrary canonical curve on f (that is, one cut out by an adjoint surface of order $m - 4$) intersects the general curve C of an irreducible system $|C|$ (one whose general curve does not break up) in a group of points which, added to a group of the characteristic series of C and to the base points of the system each counted with its degree of multiplicity, constitutes a group of the canonical series.

For each linear system $|C|$ of curves there exists a certain other linear system $|C_a|$ of particular interest called the adjoint system. A very important and beautiful property of the adjoint system is that in the case $p_g > 0$ every irreducible linear system which admits an adjoint system is contained in this adjoint, and the residual system with respect to the adjoint is the canonical system. Accordingly each system of curves on f has, with respect to its adjoint, a residual system which is independent of the particular system chosen. Conversely, if on a given surface any irreducible system of curves is contained in its adjoint, then every other irreducible system is also contained in its adjoint.

If p_g is zero, the system $|C|$ is not contained in its adjoint system $|C_a|$. But it may happen that the system $|2C|$ (defined as that system whose base points are the same as those of $|C|$ but with double the multiplicity) is contained in $|2C_a|$. In that case the same property will be possessed by every linear system of the surface. The system which is residual to $|2C|$ with respect to $|2C_a|$ is the same, whatever system $|C|$ is chosen. This residual system is called bicanonical. The number of linearly independent bicanonical curves is an invariant P called the bi-genus. The importance of the bi-genus is due to a theorem by Castelnuovo which says that the necessary and sufficient condition for a surface to be unicursal is $p_n=0$, $P=0$.

Space will not permit us to mention other interesting results contained in this part of the work. Suffice it to say that in the 200 pages devoted to surfaces and curves on them, including the long note referred to above which completes and brings up to date the most important points in this theory, we have an admirably lucid and well considered presentation in a moderate compass of the theory of algebraic functions of two variables from the algebraic-geometric point of view.

In Chapter VII, which is a reprint with scarcely any alterations of a memoir by Picard in the *Journal de Máthematiques* for 1889, we begin with the transcendental part of the subject. The double integrals of the second kind on the algebraic surface $f(x, y, z) = 0$ are here defined to be of the form

$$\int\int R(x, y, z) dx\, dy, \tag{2}$$

in which R is a rational function so constructed that when an arbitrary point A is taken on the surface, it is possible to find two rational functions U and V such that the difference between (2) and the integral

$$\int\int \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dx\, dy \tag{3}$$

remains finite in the vicinity of A. The functions U, V may vary with the point A. It is supposed that A is not at infinity. This condition can always be satisfied by making, if necessary, a suitable transformation. If A is a multiple point of f, it is always possible to divide the region about A into a number of regions each of which corresponds birationally to a simple region on another surface. If in each of these simple regions the integral has the property just described, it will be said to have the property of an integral of the second kind in the multiple point A. In order to justify and to make (2) more clearly a generalization of the familiar abelian integral of the second kind, the author remarks that it has the property of invariance relative to birational transformations. Moreover the same form of definition can be applied to the simple abelian integral $\int R(x, y)dx$. For, such an integral is of the second kind provided it is possible to find a rational function U such that the difference

$$\int R dx - \int \frac{dU}{dx} dx$$

is finite in the vicinity of a point of the given curve.

With regard to the abelian integral of the second kind, it is well known that every such integral can be expressed linearly in terms of a finite number $2p$ of such together with an additive term of the form

$$\int \frac{dU}{dx} dx.$$

The question naturally arises whether a corresponding theorem exists for the double integrals of the second kind. This is found to be the case; namely, every such integral is linearly expressible in terms of ρ_0 such integrals together with an additive term of the form (3). The demonstration of this theorem is attained as the result of a long series of transformations which, with some illustrations, forms the chief content of this chapter. The number ρ_0 is a new invariant for the surface f.

Another definition of the integral of the second kind, which is a more obvious generalization, characterizes it by the property that all its residues are zero relative to the curves C along which the rational function R becomes infinite. This definition is shown to be equivalent to the first.

In the case of a surface f with ordinary singularities the double integral of the second kind can be reduced, by the subtraction of a suitable integral, to the form $\int Q dx dy / f'_z$ in which Q is a polynomial in x, y, z of limited degree.

It is important to be able to tell when two integrals of the second kind are essentially distinct, that is, when their difference is not expressible in the form (3). This evidently reduces to the problem of recognizing when a given rational function of x, y, z is expressible in the form $\partial U/\partial x + \partial V/\partial y$. On the solution of this problem depends the exact evaluation of ρ_0. The authors attempt no more at this point than to make clear the nature of the difficulties in the way of a solution, and remark that this question is intimately connected with the theory of integrals of total differentials of the third kind.

Chapter IX accordingly takes up the theory of this class of integrals, which have already been defined in volume I. We are here confronted with a very difficult part of the theory and some important problems are left unsolved. The integrals of the third kind have logarithmic discontinuities along certain algebraic curves on f, called logarithmic curves. The fundamental theorem concerning the existence of such integrals is this: On a surface f having only ordinary singularities, one can trace $\rho - 1$ particular algebraic irreducible curves $C_1, \cdots, C_{\rho-1}$ such that there does not exist any integral of the third kind having these and only these for logarithmic curves, but such that an integral does exist having for its sole logarithmic curves any ρth curve whatever together with the curves C. The exact determination of this number ρ appears to be a very difficult matter, the solution of which is obtained in the present work only for special classes of surfaces. The lack of a complete solution of this problem is especially unfortunate on account of its connection with the double integrals of the second kind. The number ρ is invariant under birational transformations only when the correspondence does not involve fundamental points or exceptional curves.

Another problem of interest connected with these integrals is whether in case the linear connection p_1 of the surface f is unity every integral of the third kind is expressible in the form

$$(4) \qquad \sum A_k \log R_k(x, y, z) + P(x, y, z)$$

in which R_k and P are rational functions of x, y, z. There is ground for believing that such is the case. A considerable portion of this chapter is devoted to the study of various special classes of surfaces for which the integrals of the third kind assume the form (4). The result is especially simple and elegant in the case of the Kummer surface.

The tenth chapter is addressed to the difficult task of determining the number ρ_0 of the double integrals of the second kind. As already observed above, the problem reduces to the consideration of the possibility of expressing Q/f'_z in the form

$$\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}. \tag{5}$$

It is first shown that such an identity can be reduced to a form in which A and B in (5) are rational functions of x, y, z which become infinite only along the curves $C_1, \cdots, C_{\rho-1}$ previously associated with the integrals of total differentials of the third kind. It is then shown how, if such a system of curves C is once known, one may recognize when the identity in question can be satisfied and thus evaluate the number ρ_0.

These results are applicable for the effective determination of ρ_0 in a given case, but do not lead to a general law for this number. Some particular examples are given as illustrations. Thus, to mention but one, for the surface $z^2=f(x)F(y)$ in which f and F are polynomials of degree $2p+1$ and $2q+1$ respectively, ρ_0 is found to be equal to $4pq$. A very curious property of ρ_0 is brought to light by some of these cases,—its value is influenced by the arithmetic nature of the coefficients in the equation of the given surface. Thus, in case of the surface $z^2=f(x)f(y)$ in which f is a polynomial of degree 3, we have in general $\rho_0=3$. But when the coefficients in f satisfy the arithmetic conditions for the complex multiplication of the elliptic functions associated with irrationality $\sqrt{f(x)}$, then $\rho_0=2$. This peculiarity will evidently render a general formulation of laws relating to this invariant extremely difficult.

In spite of these obstacles, the task of determining a formula for ρ_0 is subsequently resumed and after a long but extremely interesting analysis we reach in Chapter XII the formula $\rho_0=N-4p-(m-1)+2r-(\rho-1)$, in which N is the class of the given surface f, m is its degree, r is the linear connection of f diminished by 1. This relation is deduced under the supposition that f has only ordinary singularities and, from the remark made above, it may fail for particular arithmetic values of the coefficients in f.

The course of investigation leading up to this formula involves a long discussion of the periods of double integrals of the first and second kinds. The properties of the linear differential equation E, introduced in volume I, is fundamental for this part of the work. This is the equation of order $2p$ whose solutions are the $2p$ periods of the abelian integral $\int Qdx/f'_z$ in which y is regarded as a parameter. Denoting by $\omega(y)$ any one of these periods, the integral $\int \omega(y)dy$ taken over a closed path in the plane of the complex variable y is a period A for the double integral $\int\int Qdxdy/f'_z$ provided that $\omega(y)$ returns to its original value after y has described the given path. The number of periods A is determined with the help of the differential equation E. The connection between the number ρ_0 and the number of periods then leads to the formula given above.

Chapter XIII is devoted to a determination of the numbers r_0 and r of linearly independent integrals of total differentials of the first and second kinds. These numbers are $r_0 = \delta$, $r = 2\delta$, in which δ is the default ω_{m-3}, all the other defaults $\omega_h(h > m - 3)$ being supposed zero. This assumption is no restriction of generality as it can always be satisfied by a suitable transformation of the surface.

The last chapter, XIV, has for its object the illustration of the previous theory by applying it to a particular class of surfaces, namely, those whose coordinates are expressible as hyperelliptic functions of two parameters. This parametric representation lends itself with facility to the calculation of the different invariant numbers which form the basis of the present theory, and leads to simple expressions for the different types of integrals.

We remark, in conclusion, that while the present work brings to light with great clearness the difficulties which surround the extension of the theory of functions beyond the case of one variable, it affords, in the brilliant achievements which it records, the confident hope that this great field will soon become familiar territory. The splendid work of M. Picard, in the highly important results it embodies and the many indications it offers of problems pressing for solution, will undoubtedly be a powerful stimulus to activity in this line.

J. I. HUTCHINSON.

SHORTER NOTICES.

Correspondance d'Hermite et de Stieltjes, publiée par les soins de B. BAILLAUD, H. BOURGET. *Avec une Préface de* ÉMILE PICARD. Vol. I, xx + 477 pp.; vol. II, vi + 464 pp. 8vo. Paris, Gauthier-Villars, 1905.

IT is all too seldom that the mathematical public gets even a furtive view of the private life of its masters, or is allowed to penetrate into their ateliers and observe how their illustrious works are elaborated. In other fields of science, in literature and art, we are more fortunate. How great would be our loss if we knew nothing of a Darwin, a Goethe, or a Michel Angelo except their finished works. It is true that mathematicians are not altogether destitute of information regarding the inner life of their leaders. Who can read the biographies of Hamilton, Abel, or Jacobi, or the letters of Gauss to Schumacher and Bessel without being touched and edified?

The present collection of letters which passed between Hermite and Stieltjes between the years 1882 to 1894 forms a very precious accession to the mathematician's scanty library of personalia, and will be welcomed by all those who love to associate with the works of a great man the man himself.

A sketch of Hermite's life has already been given in the BULLETIN,* we will therefore devote here only a few lines to the all too brief career of Stieltjes. The son of a distinguished civil engineer, Thomas-Jean Stieltjes was born at Zwolle, Holland, December 29, 1856. He studied at the Ecole Polytechnique of Delft, where his great talents were recognized both by teachers and classmates. In spite of his exceptional abilities he did not graduate, being prevented by an instinctive dislike for competition, which seems to have pursued him all his life. In 1877 he entered as computer the Observatory of Leyden and later took part in the work of observation. But soon celestial mechanics and the higher mathematics began to absorb his attention, and interfere to such an extent with his routine duties, that he entertained the very bizarre idea of giving up his position (although living then in straitened circumstances), and coming to America to study under Sylvester, who was lecturing at the Johns Hopkins University.

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* Vol. 13 (1906–7), pp. 182–190.

At this critical moment (1882) he had the good fortune to come under the beneficent influence of Hermite.* This was brought about by a letter, afterwards published in the *Comptes Rendus*, in which Stieltjes communicated to Hermite some of his results on the coefficients of the development of the perturbative function when the inclination of the orbits is considerable. Other letters followed, and soon the most friendly relations were established between the veteran Hermite and the youthful Stieltjes. His passion for mathematics now became so absorbing that Stieltjes the following year (1883) gave up his post in the observatory and devoted himself entirely to his favorite study. Numerous papers were the results of his efforts, and public recognition came in the form of the doctorate, honoris causa, from the University of Leyden, and an election to the Academy of Amsterdam.

But his hopes to an academic career received a cruel blow by his failure to obtain the vacant chair in mathematics at the University of Groningen, although the favorite candidate of the faculty. Writing to Hermite (March 13, 1884) he modestly states: "Probablement la raison aura été que n'ayant point eu l'occasion de suivre le chemin ordinaire, je n'ai point obtenu un grade à l'Université."

In 1885 Stieltjes left Holland for good, and took up his abode in Paris. He became a French citizen, and took his doctor's degree at the University of Paris the following year (June, 1886) presenting a thesis, "Sur les séries semi-convergentes." By virtue of his great talents, and supported by the powerful influence of Hermite, Stieltjes obtained at once a position as chargé de cours at the University of Toulouse, and three years later he received the professorship there. But his arduous and incessant labors began to undermine his health. In 1892 and 1893 he was obliged to pass the winters in the mild climate of Algiers, but without permanent benefit. He died the last day of the year 1894, with a brain teeming with fertile ideas and projects unfulfilled.

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The letters number somewhat more than 400, and are altogether charming both on account of the frankness and freedom of their style, and on account of the great variety of the topics discussed. A large share of the earlier letters treat of the theory of numbers, particularly those parts connected with the elliptic functions. Both Hermite and Stieltjes had a strong arithmetical bent. "Je vois aussi, Monsieur, que vous êtes un ami de l'arithmétique, et que vous partagez mon admiration pour Gauss et Eisenstein" writes Hermite in one of his first letters. Another time he writes to Stieltjes "Je confie maintenant à votre cœur arithmétique . . ." Stieltjes was much interested in Riemann's work on prime numbers and devoted considerable labor to prepare a doctor's thesis on this subject.* We therefore find an occasional discussion of Riemann's ζ function, supplemented in the appendix by four noteworthy letters to Mittag-Leffler. As is well known, Hermite first showed the transcendence of e, and a number of letters treat of this and π. In reply to Hermite's praise, Stieltjes writes with characteristic modesty "Quant à ma démonstration de la transcendence du nombre e, elle est évidemment fondée entièrement sur vos principes, et je n'ai fait que généraliser votre démonstration." Stieltjes efforts relative to π were without result and in a later letter he is "heureux d'avoir echappé au nombre π qui m'a causé un vrai cauchemar."

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* In 1892 he writes Hermite: "J'ai perdu mon père bien avant, en 1878; cette date est, pour moi, la fin d'une jeunesse heureuse; la période qui l'a suivie a été bien plus tourmentée, et pas sans difficultés."

Although both Hermite and Stieltjes were deeply interested in the higher arithmetic, by far the larger part of their correspondence relates to analysis. Naturally the theory of continued fractions, to which Stieltjes contributed so much that is fundamental, is often treated. From a letter to Hermite we know that Stieltjes began their study as early as 1877 or 1878; while for Hermite such algorithms had formed a familiar instrument of research for two score years. We can well understand the benevolent interest that the aged Hermite took in the brilliant discoveries of his young friend in this field, and it is pleasing to hear Hermite break out in such heartfelt praise as this: "Vous êtes un merveilleux géomètre, les recherches nouvelles, sur les fractions continues algébriques que vous me communiquez sont un modèle d'invention et d'élégance; ni Gauss, ni Jacobi ne m'ont jamais causé plus de plaisir."

A great deal of space is occupied in discussing Legendre's functions of the first and second kind, questions of development, asymptotic expressions, roots of $Q_n(x)$, etc.

Klein's article of 1890 on the roots of the hypergeometric function at once attracted Stieltjes' attention and he writes

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* He afterward selected another subject, as stated above.

Hermite: "La démonstration de M. Klein est extrêmement intéressante, elle repose entièrement sur la conception des feuilles multiples de Riemann et des considérations géométriques. Cela prouve bien l'utilité de ce mode de représentation, mais je vous dirai que la géométrie y intervient trop, à mon gout. . . . Dès que j'ai vu l'énoncé du théorème je n'ai pu m'empêcher de penser qu'on pourrait y arriver par la méthode de Sturm en s'appuyant sur les relations entre les fonctions F contiguës, etc." Hermite replies "Mais la communication extrêmement intéressante que vous m'avez faite du beau théorème de M. Klein . . . m'arrache à la torpeur qui m'a envahi depuis plusieurs jours. Je vous chicanerai au sujet de votre prétention de le démontrer par la méthode de Sturm . . . Il me semble que cette méthode n'a jamais été appliquée et n'est applicable. . ." Whereupon Stieltjes in his next letter playfully begins to develop his method with: "Maintenant, permettez moi de défendre mon idée de démontrer le théorème de M. Klein à l'aide de la méthode de Sturm." A subject of unfailing interest especially in the later letters is the Γ function in all its astonishing and intricate relations. Mechanical quadrature and asymptotic developments are also favorite topics.

Thus we might go on describing these delightful mathematical causeries which follow one another in rapid and unaffected succession, but we hope that the above will give an idea of the rare treat these letters will afford.

JAMES PIERPONT.

Cartesian Plane Geometry. Part I.: *Analytical Conics.* By C. A. SCOTT, D.Sc. London, J. M. Dent and Company, 1907.

IF this most excellent book is really meant to be a text-book for schools, we are afraid that some one will now have to make a school for the text-book. If it is meant for the libraries of teachers and prospective teachers, it is as near perfection as one could wish; full of the best ways of doing things, of excellent examples and of inspiration for every real teacher. So, whatever faults we may see in the book are those which arise when we consider it as a function of the things it is meant to be as a text-book.

The book, rather unprepossessing in its outward appearance, contains 428 pages, a great number of illustrative examples, and 700 problems. Its thirteen chapters, which of course are a continuous development of the theory of conics, are headed as follows: Introduction, Coordinates of point and line, Representation of point and line by equations, Loci and envelopes, Conics, Relation of straight lines and curves, Tangent at a point and polar properties, Bisected chords and diameters, Asymptotes, Properties of conics (summary), Change of axes, Systems of conics and miscellaneous examples.

The distinctive feature of the book is the use of line coordinates throughout. This, as the introduction states, makes many problems easier, and immensely increases the power of the subject. Geometrically, the line as element and line envelopes are as simple as the point and point loci, and it seems too bad that the unfortunate leaning of our ancestors toward the use of a point to pick their hieroglyphics and draw their pictures, should have led us to be almost void of visualization of how curves must look if their tangents satisfy certain conditions.

Other features of the book are the introduction of the ideal elements by means of ratios defining a point on a line; the use of the circular points and null lines with their relations to foci; the power of a point with respect to a curve; abridged notation; the reality of the imaginary elements; centers of curvature and evolutes; graphical solutions of equations and the determination of conics cutting a circle in a regular polygon. A great deal is said on the subject of bisected chords and diameters and on the properties of configurations arising from conormal points on a conic; in fact the long list of miscellaneous examples worked out in the last chapter has many of these problems which would be the despair of the ordinary undergraduate, *e. g.*, the problem of finding the centroid and orthocenter of a circumscribed triangle of a conic whose points of contact are conormal covers four pages of rather tedious algebra.

Space will permit mention here of but a few of the admirable arrangements and developments of this book.

First of all is the use of line coordinates to solve those problems for which they are best adapted. In this way the student is given another weapon with which to attack the subject; however, Professor Scott often dulls it for him, by the apologetic way in which she uses it, for she usually follows a simple demonstration with the use of line coordinates by a difficult one in point coordinates.

The condition of incidence of a point and line is derived before anything is said about the equation of a point or line, which thus arises naturally. At the beginning of the book various types of coordinates are used, though no examples throughout the entire book are solved by the use of polar coordinates, and thus the student reader will fail to appreciate their power.

A good idea is the derivation in general of the equation of a conic without regard to axes

$$(x-p)^2+(y-q)^2=\frac{e^2}{l^2+m^2}(lx+my+n)^2$$

and hence arrival at once, by a proper choice of axes, which is seen analytically, at simple canonical forms for the different conics.

The general treatment of ideal and imaginary elements on the same basis as those ordinarily called real ones is well illustrated by the treatment of asymptotes as tangents at infinity.

Parametric equations for certain problems in conics and straight lines are used, and although in many instances these are introduced only to be got rid of, yet their importance is emphasized more than in some books. It seems strange that someone does not thoroughly develop this extremely important type of equations, in an elementary text-book.

The line equation of the general conic is introduced very cleverly, as follows: First the equations of the lines through the origin and the points of intersection P_1, P_2 of the conic with the line $\xi x+\eta y+1=0$ are found by means of the necessary condition of homogeneity and the abridged notation; then the condition that these lines through the origin and hence P_1, P_2 be coincident is imposed and thus the condition that $\xi x+\eta y+1=0$ be a tangent is found.

The idea of the graph of an equation seems to be greatly neglected; in fact the whole book gives one the impression that the mere picture of the equation has but slight importance in an analytic geometry. So we find very few examples of point graphs and practically none of line loci. As a natural sequence of this point of view Professor Scott omits the discussion of an equation, as such. This seems like a defect to us, both from the pedagogic and practical standpoint, for in few other ways does such simple analysis yield such good results.

As the book is essentially analytic, all pure geometry examples are put in the background. We cannot agree with this way of presenting the subject to the immature student, for we believe that very few of the latter ever become interested at the beginning in geometry which is only analytic.

The subject of change of axes is rather unsatisfactorily treated. In the first place it is left almost to the end of the book, and thus the student is deprived of the use of a powerful tool for simplifying his problems and for avoiding what Professor Scott designates "algebraic gymnastics." Secondly the formulas of transformation are unrigorously derived, holding necessarily only for the special figures of the book. Finally all examples are worked by comparison with the general case *i.e.*, the general equation of the conic is transformed and its new coefficients found and these are used for the special cases. This undoubtedly gives good practical results; but the student would soon lose sight of what he was doing and the raison d'être of his formulas.

One other fault of the book appears to us to be that as a textbook it contains so many ideas and methods that it would be confusing to any student; *e. g.*, under diameters and reciprocal polars the same examples are worked in detail in three different ways. This makes the book invaluable for our libraries; but if we judge the young student aright, the study of these different methods would tend to confuse what ideas he might successfully gather from the most simple and hence best method of attacking the problem.

E. GORDON BILL.

Grundlagen der Geometrie. Von DAVID HILBERT. Dritte Auflage. Leipzig, Teubner, 1909. vi + 279 pp.

THE present edition of Hilbert's celebrated treatise on geometry appears as a seventh volume in a series of essays, "Wissenschaft und Hypothese," whose first volume is a translation of Poincaré's well known book bearing the latter title. It differs but little from the second edition. Two reprints of papers by Hilbert, "Ueber den Zahlbegriff" * and "Ueber die Grundlagen der Logik und Arithmetik," † appear as appendices VI and VII respectively; also additional references to investigations of other authors are inserted.

Since very full consideration has been given Hilbert's book in the BULLETIN ‡ and elsewhere § the reviewer must refrain from any lengthy discussion. Some of the logical phases of Hilbert's axioms are instructively pointed out in a controversy between Frege and Korselt in the *Jahresbericht der Deutschen Mathematiker-Vereinigung.*[*]

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* *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 8 (1900).

† Verhandlungen des III Internationalen Kongresses in Heidelberg, 1904.

‡ See vol. 6, p. 287; vol. 9, p. 158; vol. 10, p. 1.

§ Cf. E. B. Wilson, *Archiv der Mathematik u. Physik* (3), vol 6 (1904), p. 104.

A. R. SCHWEITZER.

Vorträge über den mathematischen Unterricht an den höheren Schulen. KLEIN-SCHIMMACK. Teil I. *Von der Organisation des mathematischen Unterrichts.* Leipzig, 1907, pp. ix + 236.

PROFESSOR KLEIN has for years been calling attention to the fact that it is the duty of the universities to furnish instruction not only in the subject matter of mathematics but also in the questions relative to the teaching of the subject, since teaching is the profession to which the great majority of the students are looking forward; and in his own university (Göttingen) he has admirably been putting into practice what he preaches. The present volume makes accessible to the wider public the substance of a course of lectures thus delivered at Göttingen in 1904–5.

It is a work that at once commanded thoughtful attention in all quarters of the nation for which it was especially intended, but it also deserves and will receive careful study far beyond the national confines.

The work is divided into the following sections, whose titles give a general idea of its plan and scope:[†]

Introduction, pages 1–9; I. Elementary schools, 10–18; II. The six lower classes[‡] of the (boys') higher schools, 19–43; III. Girls' schools and trade schools, 44–66; IV. The historical development of instruction in mathematics in the German higher schools, 67–99; V. The three upper classes[§] of the higher schools according to the curricula of 1901, 100–126; VI. Propositions for reform in the upper classes of the higher schools, 127–159; VII. The universities and technical schools, 158–189; Conclusion, 189–190; Appendix (containing reprints of two of Klein's earlier papers and of the curriculum proposed by the Commission at Meran *), 191–236.

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[*] See vol. 12 (1903), pp. 319, 368, 402; vol. 15 (1906), pp. 293, 377, 423; vol. 17 (1908), p. 98.

[†] With each numbered title, except No. IV, the words "mathematics in" are of course implied.

[‡] In mathematics these six classes correspond roughly to our grades 4 to 8, and the first year of the high school.

[§] In mathematics these classes correspond roughly to the second and the third year of the high school and the freshman year in college.

The form of the work is, as already stated, that of lectures to students of a German university preparing to teach mathematics in Germany. It therefore naturally contains much matter that relates exclusively to German schools and much of a more general character that is expressed in the terminology of the German school system; but there is also not a little, and that of the most important and characteristic, which is intelligible without any acquaintance with the details of the German organization.† The chapters of widest general interest are Nos. II, V, and VI.

As to general methods, Klein finds that the genetic method has been steadily gaining ground and has entirely displaced the unduly formalistic systems of earlier days. In this connection, the question is raised, pertinent also elsewhere than in Germany, whether "many a university lecture course might not take a lesson in pedagogic-psychologic preparation from the modern types of instruction in the schools" (page 24).

Quite a little space is devoted to a discussion of the function concept, which, in geometric form, is to be made the *soul* of the mathematical courses in the schools (page 34). In close connection with this lies the question of the interrelations in time and subject matter of the various branches of mathematics taught in the schools, a question of prime importance for us Americans, since in this matter we are far behind the Germans, who have long been teaching algebra (including numerical work) and geometry (with trigonometry) side by side, details being left in the main to the discretion of the teacher. Let us hear whether Klein is satisfied with this, or whether perchance he even believes that the Germans have gone too far in this direction.

"According to my feeling, instruction in mathematics is at present still conducted with too much external separation of algebra and geometry, and with this external separation a deeper one often goes hand in hand. The principle of purity of methods may be interesting in the higher special courses of the university, but unfortunately it is also sometimes proclaimed in the schools. 'Geometria geometrice'; 'No numbers in geometry'; 'Happily, algebra does not need geometric crutches'; 'By all means, no diagrams in arithmetic'; such are the mottoes for the sacrifice of thorough treatment to the higher honor of purity of method. It is needless to discuss the unpedagogic character of this position. The secondary schools, whose ideal aim is general culture, should evidently present mathematics as an organism whose various parts stand in active and vital interrelations.

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*See BULLETIN, vol. 12 (1906), pp. 347–352

† Various descriptions of the German secondary schools, quite sufficient to make the present work intelligible throughout, are within reach of the American reader. For example:

Russell: German Higher Schools. New York (Longmans). 1898.

Bolton: The Secondary School System of Germany, New York (Appleton), 1900.

Young: The Teaching of Mathematics in Prussia. New York (Longmans), 1900.

"If, gentlemen, any mathematical question presents itself to you in later life you will have no leisure to ask: 'Shall I proceed purely geometrically, perhaps even projectively, avoiding all metric concepts, or shall I follow the method of the pure theory of numbers?' No; the pressing thing will be to be able to help yourself out on the basis of the totality of your mathematical acquisitions. And this will in general be easier the closer together the different forms of mathematical thought have been brought in your instruction. Just on that account it will be well if the arithmetic, algebra, and geometry of the lowest classes coalesce in 'Obertertia' and 'Untersecunda' (ages 13–15) and culminate in a grasp of the function concept. Of course, geometry is not to be sacrificed to algebra or vice versa; by mutual assistance, both forms of mathematical thinking will not lose but gain."

The fifth chapter is devoted to a discussion of the course in mathematics in the three upper classes, the last of which takes up approximately our freshman work in college. The list of subjects prescribed by the curricula for these years is taken up in detail (Algebra, pages 101–109, Geometry, pages 121–126), and the considerations are interesting and suggestive to Americans also. A large portion of the chapter (pages 109–121) is devoted to an "Excursus on the question of the infinitesimal calculus in the schools." So far as the *last* school year (our freshman year) is concerned, the reasons so cogently stated would find ready acceptance by some in America. The writer has for over a decade given an introductory course in calculus for freshmen, and no doubt the ideas of the calculus have been more or less extensively introduced into freshmen work in numerous instances. Mention should be made especially of the work in mathematics of the first year in the Massachusetts Institute of Technology, as set forth in the excellent and suggestive work of Woods and Bailey,* which aims to organize college algebra, plane analytic geometry, and the elements of the calculus into a whole, whose unifying note shall be *mathematics* rather than any one of the subjects named. But to carry the calculus back into precollegiate work, even in the most elementary way, would not at present be practicable with us. Still Klein's proposal to introduce the ideas of derivative and integral in what corresponds roughly to our third high school year may well be feasible in Germany (where the pupils remain in the same school, and probably with the same teacher during the years in question), and has there met with strong, though by no means unanimous, support.

Mere mention must suffice for the interesting discussion of proposed reforms in Chapter VI, and the sketch (Chapter VII) of mathematics at the universities, especially Göttingen, with its graphic representation in colors of the galaxy of famous men that in unbroken sequence have justly constituted Göttingen's pride.

The work is written in an informal, clear, and interesting style, and should find a considerable circle of readers on this side of the Atlantic.

Upon the heels of the work discussed above there has followed a lithographed volume of a course of lectures by Klein (in 1907–8 †) treating topics related to elementary mathematics from the higher standpoint, along the lines of the synoptic course recommended by the German Commission in 1907.‡ This is a further valuable and stimulating fruit of Klein's attitude mentioned at the outset, and merits attention and study by a wide circle of readers. A second volume dealing with geometric topics is to follow quite shortly, and a full review of both will appear in the BULLETIN in due course.

J. W. A. YOUNG.

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* A Course in Mathematics, vol. I., Boston (Ginn), 1907.

† Elementar-Mathematik vom höheren Standpunkte aus. Teil I: Arithmetik, Algebra, Analysis. Leipzig (Teubner), 1908, pp. 590.

‡ See BULLETIN, vol. 15 (1909), p. 262.

CORRECTION.

The following errata in the present volume of the BULLETIN have come to the attention of the editors:

Page 162, lines 8–9 relate to Dr. Irwin's paper, No. 6 on the preceding pages.

Page 324, lines 2–3, read "Note on Laplace's equation," read October 25, 1902.

NOTES.

BEGINNING July 1, 1909, the publication of the *Bulletin* and the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, heretofore in charge of The Macmillan Company, will be taken over by the Society. Orders and subscriptions of non-members should be sent direct to the Society at 501 West 116th Street, New York, N. Y.

A FEW copies of the reprint of the Chicago Symposium on mathematics for engineering students are still on hand and may be procured from the Secretary of the Chicago Section, Professor H. E. Slaught, University of Chicago.

AT the meeting of the London mathematical society held on May 13, the following papers were read: By H. W. TURNBULL, "Ternary quadratic types"; by J. G. LEATHEM, "The theorem of Gauss in the theory of attraction"; by J. E. LITTLEWOOD, "Criteria for continuity and discontinuity of a function defined by an infinite product."

The committee on a syllabus of geometry for secondary schools recently appointed by the Council of the American federation of teachers of the mathematical and natural sciences in connection with the mathematics section of the department of secondary education of the National education association has been arranged as follows: Professors H. E. Slaught, chairman, C. L. Bouton, F. Cajori, H. E. Hawkes, E. R. Hedrick, H. L. Rietz, D. E. Smith, Messrs. Wm. Betz, E. L. Brown, W. B. Carpenter, W. W. Hart, F. K. Newton, E. R. Smith, R. L. Short, and Miss Mabel Sykes.

ON the occasion of the second centenary of the birth of Leonhard Euler, in 1907, the Swiss society of naturalists appointed a committee to investigate and report on the feasibility of publishing all his memoirs. The fourth international congress of mathematicians, in Rome, 1908, passed a resolution commending the action of the Swiss society, and urging academies, mathematical societies, and individual mathematicians to assist in the undertaking. On August 30, 1908, the committee reported its readiness to undertake the task, provided sufficient financial assistance could be secured. Before making a public request, the committee first addressed itself to the Swiss people, from whom it has received about 100,000 francs. At the Cologne meeting of the Deutsche Mathematiker-Vereinigung the sum of 5,000 francs was appropriated for the same purpose. The Paris academy of sciences has subscribed for forty complete sets. It is estimated that the total cost of publication will be 400,000 francs; the complete works will comprise about forty volumes, and are offered at subscription at not to exceed 25 francs per volume, to be paid upon appearance of the single volumes.

With this much already accomplished, the society now addresses itself to the world, with the special request that academies, mathematical societies, physical societies, associations of engineers and of industrial undertakings follow, so far as they can, the example of the organizations mentioned. In particular, libraries of institutions of learning, of insurance companies, and of cities are invited to send in their subscriptions at once. Moreover, voluntary contributions from individuals will be of great assistance to the committee. All subscriptions and contributions should be sent to Professor Ferdinand Rudio, Zürich V, Dolderstrasse 111.

THE eighty-first convention of German naturalists and physicians will be held at Salzburg, Austria, September 19–25, 1909. The Deutsche Mathematiker-Vereinigung will hold its annual summer meeting in affiliation with it, under the presidency of Professor M. KRAUSE, of the technical school at Dresden. Titles and abstracts of papers to be read at this meeting should be sent to the secretary, Professor A. KRAZER, Westendstrasse 57, Karlsruhe, as soon as possible. Reports on algebraic functions of several variables and the theory of integral equations will be presented.

WITH the new editorial organization of the *American Mathematical Monthly* the University of Illinois joins the University of Chicago in assisting in its financial support. The *Monthly* does not desire to compete with other mathematical periodicals, but hopes to offer an opportunity for less mature writers to express views on points of interest and to publish original solutions, etc., which are of value, yet hardly appropriate for more ambitious journals.

THE following courses in advanced mathematics are announced for the academic year 1909–1910:

UNIVERSITY OF GÖTTINGEN (first semester). — By Professor F. KLEIN: Projective and non-euclidean geometry, four hours; Seminar, two hours. — By Professor D. HILBERT: Partial differential equations, four hours; Seminar, two hours. — By Professor C. RUNGE: Mechanics, four hours; Seminar, two. hours. — By Professor L. PRANDTL: Thermodynamics, two hours; Seminar, two hours. — By Professor E. LANDAU: Theory of functions, four hours; Theory of prime ideals, one hour. — By Professor W. VOIGT: Electrodynamics, four hours. — By Professor E. ZERMELO: Logical foundations of mathematics, two hours. — By Professor F. BERNSTEIN: Calculus of probabilities, two hours. — Fourier's series, two hours. — By Dr. O. TOEPLITZ: Determinants, with applications to analytic geometry of space, four hours. — By Dr. P. KOEBE: Algebra, four hours; Differential equations, two hours.

INDIANA UNIVERSITY. — By Professor S. C. DAVISSON: Theory of surfaces, three hours; Differential equations, three hours (*a*, *w*). — By Professor D. A. ROTHROCK: Theory of functions, three hours (*a*, *w*); Advanced calculus, three hours; History of mathematics, three hours (*s*). — By Professor U. S. HANNA: Invariants, two hours. — By Professor C. HASEMAN: Applications of partial differential equations, three hours. (*a* = autumn, *w* = winter, *s* = spring.)

INDIANA UNIVERSITY (summer quarter, June 24 to September 3, 1909). — By Professor D. A. ROTHROCK: Theory of equations, five hours; Advanced calculus, five hours. — By Professor U. S. HANNA: Theory of numbers, five hours. — By Professor C. HASEMAN: Vector analysis, five hours.

UNIVERSITY OF PENNSYLVANIA. — By Professor E. S. CRAWLEY: Solid analytic geometry, two hours; Higher plane curves, three hours; Mathematics of insurance, two hours. — By Professor G. E. FISHER: Advanced calculus, two hours;

Calculus of variations, two hours. — By Professor I. J. SCHWATT: Infinite series and products, two hours; Definite integrals, three hours. — By Professor G. H. HALLETT: Modern higher algebra, three hours (first half year); Galois theory of equations, three hours (second half year); Theory of groups of a finite order, three hours; Lie's theory of continuous groups, three hours (first half year). — By Professor F. H. SAFFORD: Mathematical theory of precision of measurements, three hours (first half year); Curvilinear coordinates, three hours (second half year). — By Dr. O. E. GLENN: Invariants and covariants, three hours.

THE Belgian academy of sciences announces the following prize subjects for 1910:

A systematic and didactic treatise on recent advances in the theory of partial differential equations of the second order is desired. Prize, 800 francs.

Analyze and complete the investigations made in the calculus of variations since 1850. Prize 600 francs.

Manuscripts should be written in French or in Flemish, and submitted to the secretary of the academy under the usual conditions before August 1, 1910.

THE Jablonowski society of Leipzig announces the following prize problem for the year 1911:

"It is desired that the theory of the rainbow should be extended, especially the distribution of light over the surface of a sphere, with a stated degree of approximation. The diameter of the sphere should be small enough to avoid bending of the spectrum bands, and large enough to be at least comparable with the length of light waves."

PROFESSOR G. LORIA, of the University of Genoa, has been elected foreign member of the Bohemian academy of sciences of Prague.

PROFESSOR T. LEVI-CIVITA, of the University of Padua, and Professor C. ARZELÀ, of the University of Bologna, have been elected corresponding members of the mathematical society at Charkow.

SINCE the complete destruction of the University of Messina by the earthquake, various professors have been temporarily assigned to duties in technical schools, academic institutes, and other schools. Professor T. BOGGIO has been appointed professor of mechanics at the institute of Florence.

Professor C. Carathéodory, of the University of Bonn, has accepted a professorship of mathematics at the technical school at Hanover.

Professor J. Hadamard, of the University of Paris, has been appointed professor of analytic mechanics at the Collège de France, as successor to the late M. Lévy.

Professor M. Kutta, of the technical school at Munich, has accepted a full professorship of applied mathematics at the University of Jena.

Mr. J. F. Cameron has been appointed tutor in Gonville and Caius College, Cambridge.

Professor W. L. Tanner, of the University of Monmouthshire, South Wales, has resigned after holding the chair of mathematics for twenty-six years.

Professor G. H. Ling, of Columbia University, has been appointed head of the department of mathematics in the newly established University of Saskatchewan.

At the University of Minnesota, Dr. A. L. Underhill, of the University of Wisconsin, and Professor G. P. Paine, of Ripon College, have been appointed assistant professors of mathematics. Mr. J. S. Mikesh, of the University of Colorado, has been appointed instructor in mathematics. Mr. R. R. Shumway has been granted a year's leave of absence for further study. Professor G. A. Bauer will return from Europe this summer to resume his position.

At Yale University Dr. W. R. Longley has been promoted to an assistant professorship of mathematics. Dr. L. I. Hewes has resigned his position as instructor in mathematics.

The following changes have taken place at Princeton University: Professor J. H. Jeans has resigned to accept a professorship of mathematical physics at the University of Cambridge; Professor L. P. Eisenhart has been promoted to a full professorship of mathematics; Professor C. E. Stromquist has resigned, having accepted the professorship of mathematics at the University of Wyoming.

Professor F. S. Woods, of the Massachusetts Institute of Technology, has been granted leave of absence for the academic year to study in Europe. Dr. A. J. Lennes returns to the Institute as instructor in mathematics.

AT the University of Colorado Professor S. EPSTEEN has been appointed Secretary of the Graduate School. Professor Epsteen has also been elected president of the Colorado mathematical society.

THE following instructors in mathematics have been appointed at the University of Wisconsin: Dr. A. DRESDEN, in the department of letters and science: Dr. H. E. BUCHANAN, Mr. E. E. MOOTS, and Mr. E. H. MOULTON in the school of engineering.

PROFESSOR C. S. SLICHTER, of the University of Wisconsin, has been granted leave of absence during the academic year 1909–1910 to study in Europe.

PROFESSOR O. BIERMANN, of the German technical school at Brünn, died April 28, at the age of 51 years.

PROFESSOR P. MUTH, the author of the Elementarteiler, died April 30 at Osthofen, Rheinhessen, at the age of 48 years.

PROFESSOR G. VAILATI died at Rome, May 15, at the age of 46 years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (W. R.). Récréations mathématiques et problèmes des temps anciens et modernes. Vol. III. Paris, Hermann, 1909. 8vo. Fr. 5.00

BOREL (E.). Eléments de la théorie des probabilités. Paris, Hermann, 1909. 8vo. 6 + 200 pp. Fr. 6.00

BJRGGRAF (G.). Die Γ-Function für komplexe Argumente. Brünn, 1909. 8vo. 33 pp.

D'ADHÉMAR (R.). L'équation de Fredholm. Problèmes de Dirichlet et de Neumann. Paris, Hermann, 1909. 8vo. Fr. 3 50

DZIOBEK (O.). Lehrbuch der analytischen Geometrie. Teil I: Analytische Geometrie der Ebene. 2te verbesserte Auflage. Braunschweig, Graff, 1909. 8vo. 8 + 358 pp. M. 4.50

HASS (P.). Zur Definition des Begriffs der eindeutigen analytischen Funktion. (Diss.) Kiel, 1908. 8vo. 55 pp.

JORDAN (C.). Cours d'analyse de l'école polytechnique. 3e édition, revue et corrigée. Vol. I. Paris, Gauthier-Villars, 1909. 8vo. 16 + 621 pp. Fr. 17.00

KAMBLY und ROEDER. Elemente der analytischen Geometrie der Ebene. Neu bearbeitet von A. Thaer. Breslau, Hirt, 1909. 8vo. 121 pp. M. 1.30

KOJALOWITSCH (B.). Lectures upon higher mathematics. (Russian.) St. Petersburg, 1909. 8vo. 376 pp. M. 9.00

MARGOSSIAN (A.). De l'ordonnance des nombres dans les carrés magiques impairs. Paris, Hermann, 1909. 8vo. 140 pp. Fr. 2.00

MELLOR (J. W.). Higher mathematics for chemistry and physics. London, 1909. 8vo. 664 pp. Cloth.

MITTAG-LEFFLER (F.). Niels-Henrik Abel. Paris, Hermann, 1909. 8vo. 68 pp. Fr. 2.00

PICARD (E.). Traité d'analyse. Vol. 3. 2e édition. Paris, Gauthier-Villars, 1908. 8vo. 7 + 607 pp. Fr. 18.00

PFITZNER (P.). Ueber den didaktischen Wert zusammenhängender Aufgabengruppen im mathematischen Unterricht. Dresden, 1909. 4to. 32 pp. M. 1 20

SCHONER (E.). Studien zur Wurflinie. Erlangen, 1909. 8vo. 67 pp. M. 2.40

TANNERY (J.). Elemente der Mathematik. Deutsch von P. Klaess. Leipzig, Teubner, 1909. 8vo. 12 + 339 pp. M. 7.00

VEGAS (M.). Elementos de geometria analitica. Barcelona, Gili, 1909. 8vo. 240 pp. P. 4.00

WALEK (K.). Binäre kubische Transformation und Complexe. (Diss.) Munchen, 1908. 8vo. 39 pp.

WINKLER (J.). Die Schwingungsfiguren in analytischer Behandlung. Celle, 1909. 8vo. 32 pp. M. 1.50

II. ELEMENTARY MATHEMATICS.

ECKHARDT (E.). Zurückführung der sphärischen Trigonometrie auf die Geometrie des ebenen Kreisvierecks. Neue Grundlegung für die Formeln der sphärischen Trigonometrie. Leipzig, Teubner, 1909. 8vo. 6 + 155 pp. M. 4.40

GEOMETRY and graphic algebra. London, Wyman, 1909. 1 d.

GIROD (F.) et WEILL (E.). Eléments de géométrie à l'usage des élèves du premier cycle des lycées et collèges et des classes de première et seconde A et B conformes au programme du 27 juillet, 1905. Paris, André, 1909. 8vo. 399 pp.

KAMBLY und ROEDER. Stereometrie. Neu bearbeitet von A. Thaer. 32ste Auflage, Breslau, Hirt, 1909. 8vo. 167 pp. Cloth. M. 2.00

PITMAN's guide to intermediate and advanced scholarship examinations with answers to the mathematical sections for students preparing for scholarship examinations. London, Pitman, 1909. 12mo. 140 pp. 2s.

ROMERO RUBIERA (A.). Elementos de trigonometria. Alicante, López, 1909. 8vo. 77 pp. P. 1.00

THIEME (H.). Die Elemente der Geometrie. Leipzig, Teubner, 1909. 8vo. 12 + 394 pp. Cloth. M. 9.00

III. APPLIED MATHEMATICS.

Boulanger (A.). Hydraulique générale. Vol. II : Problèmes à singularités et applications. Paris, Doin, 1909. 12mo. 7 + 299 pp.

Contaldi (P.). La meccanica nella scuola e nell' industria. Vol. II. 2a edizione, riveduta ed ampliata. Milano, Hoepli, 1909. 8vo. 15 + 735 pp. L. 16.00

Cosserat (E.) et (F.). Théorie des corps déformables. Paris, Hermann, 1909. 8vo. 6 + 228 pp. Fr. 6.00

Matisse (G.). Le principe de la conservation de l'assise et ses applications. Paris, Hermann, 1909. 8vo. 65 pp. Fr. 2.50

Nelting (R.). Der nautisch-astronomische und Universal-Rechenstab zur mechanischen Lösung aller mathematischen Probleme, unter Berucksichtigung der terrestrischen und astronomischen Navigation. Hamburg, Friederichsen, 1909. 8vo. 6 + 67 pp. M. 4.00

Richarz (F.). Anfangsgründe der Maxwellschen Theorie verknüpft mit der Elektronentheorie. Leipzig, Teubner, 1909. 8vo. 9 + 246 pp. M. 7.00

Weisse (E.). Anwendung der elliptischen Funktionen auf ein Problem aus der Theorie der Gelenkmechanismen. Rostock, 1909. 8vo. 102 pp. M. 2.50

EIGHTEENTH ANNUAL LIST OF PAPERS.

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF THEIR PUBLICATION.

Ames, L. D. A Simpler Proof of Lie's Theorem for Ordinary Differential Equations. Read (Chicago) Apr. 9, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 8, pp. 384–386; May, 1909.

Bill, E. G. The Construction of a Space Field of Extremals. Read Dec. 30, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 8, pp. 374–378; May, 1909.

Birkhoff, G. D. Boundary Value and Expansion Problems of Ordinary Linear Differential Equations. Read (Chicago) Mar. 30, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 4, pp. 373–395; Oct., 1908.

—— Existence and Oscillation Theorem for a Certain Boundary Value Problem. Read (Chicago) Jan 1, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 259–270; Apr., 1909.

Bliss, G. A. See Mason, M.

Bôcher, M. On the Regions of Convergence of Power Series Which Represent Two-Dimensional Harmonic Functions. Read Oct. 25, 1902. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 271–278; Apr., 1909.

Bolza, O. The Determination of the Conjugate Points for Discontinuous Solutions in the Calculus of Variations. Read (Chicago) Mar. 30, 1907. *American Journal of Mathematics*, vol. 30, No. 3, pp. 209–221; July, 1908.

—— Heinrich Maschke: His Life and Work. Read Sept. 10, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 2, pp. 85–95; Nov., 1908.

Buchanan, H. E. On Certain Determinants Connected with a Problem in Celestial Mechanics. Read (Chicago) Jan. 2, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 5, pp. 227–231; Feb., 1909.

Carmichael, R. D. A Table of the Values of m Corresponding to Given Values of $\phi(m)$. Read (Chicago) Mar. 30, 1907. *American Journal of Mathematics*, vol. 30, No. 4, pp. 394–400; Oct., 1908.

—— Even Multiply Perfect Numbers of Five Different Prime Factors. Read Apr. 27, 1907. *Bulletin of the American Mathematical Society*, vol. 15, No. 1, pp. 7–8; Oct., 1908.

—— On the Geometric Properties of Quartic Curves Possessing Fourfold Symmetry with Respect to a Point. Read Oct. 26, 1907. *Annals of Mathematics*, Ser. 2, vol. 10, No. 2, pp. 81–87; Jan., 1909.

—— On the General Tangent to Plane Curves. Read (Chicago) Apr. 18, 1908. *American Mathematical Monthly*, vol. 15, Nos. 8–9, pp. 153–157; Aug.–Sept., 1908.

—— On Plane Algebraic Curves Symmetrical with Respect to Each of Two Rectangular Axes. Read (Chicago) Apr. 18, 1908. *American Mathematical Monthly*, vol. 15, No. 10, pp. 175–177; Oct., 1908.

—— Notes on the Simplex Theory of Numbers. Read Oct. 31, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 5, pp. 217–223; Feb., 1909.

—— On the Theory of Functions of a Triple Variable. Read Oct. 31, 1908. *American Mathematical Monthly*, vol. 16, No. 3, pp. 41–46; Mar., 1909.

CARVER, W. B. The Quadric Spreads Connected with the Configuration $\Gamma^{n+2}_{n+4,n}$, and a Special Case in the Pascal Hexagram. Read Dec. 28, 1906, and Sept. 5, 1907. *American Journal of Mathematics*, vol. 31, No. 1, pp. 1–17; Jan., 1909.

—— Degenerate Pencils of Quadrics Connected with $\Gamma^{n+2}_{n+4,n}$ Configurations. Read Dec. 31, 1908. *Bulletin of the American Mathematical Society*, vol. 15, No. 10, pp. 483–486; July, 1909.

COBLE, A. B. An Application of the Form Problems Associated with Certain Cremona Groups to the Solution of Equations of Higher Degree. Read Dec. 27, 1907. *Transactions of the American Mathematical Society*, vol. 9, No. 4, pp. 396–424; Oct., 1908.

—— Symmetric Binary Forms and Involutions. Read Dec. 31, 1908. *American Journal of Mathematics*, vol. 31, No. 2, pp. 183–212; Apr., 1909.

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—— On Commutative Linear Groups. Read Sept. 6, 1907, and (Chicago) Jan. 1, 1908. *Quarterly Journal of Pure and Applied Mathematics*, vol. 40, No. 2–3, pp. 167–196; Jan.–May, 1909.

—— On Families of Quadratic Forms in a General Field. Read (Chicago) Dec. 30, 1907. *Quarterly Journal of Pure and Applied Mathematics*, vol. 39, No. 4, pp 316–333; July, 1908

—— Definite Forms in a Finite Field. Read Sept. 10, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 1, pp. 109-122; Jan., 1909.

—— Rational Reduction of a Pair of Binary Quadratic Forms; their Modular Invariants. Read Sept. 10, 1908. *American Journal of Mathematics*, vol. 31, No. 2, pp. 103–146; Apr., 1909.

—— On the Last Theorem of Fermat. Read Sept. 11, 1908. *Messenger of Mathematics*. vol. 38, Nos. 1–2, pp 14–32; May–June, 1908, and *Quarterly Journal of Pure and Applied Mathematics*, vol. 40, No. 1, pp. 27–45; Oct., 1908.

—— General Theory of Modular Invariants. Read (Chicago) Jan. 1, 1909. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 123–158; Apr., 1909.

—— On the Representation of Numbers by Modular Forms. Read (Chicago) Jan. 2, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 7, pp. 338–347; Apr., 1909.

DINES, L. L. A Method of Investigating Numbers of the Forms $6^k.s \pm 1$. Read (Chicago) Apr. 17, 1908. *Annals of Mathematics*, ser. 2, vol. 10, No. 3, pp. 105–115; Apr., 1909.

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FORD, W. B. A Set of Criteria for the Summability of Divergent Series. Read (Chicago) Apr. 9, 1909. *Bulletin of the American Mathematical Society*, vol. 15, No. 9, pp. 439–444 ; June, 1909.

GRANVILLE, W. A. Duality in the Formulas of Spherical Trigonometry. Read Dec. 31, 1908. *American Mathematical Monthly*, vol. 16, No. 3, pp. 47–53 ; Mar., 1909.

GRIFFIN, F. L. On the Law of Gravitation in the Binary Systems. Read Dec. 28, 1906, and Sept. 6, 1907. *American Journal of Mathematics*, vol. 31, No. 1, pp. 62–85 ; Jan., 1909.

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HARTWELL, G. W. Plane Fields of Force Whose Trajectories are Invariant under a Projective Group. Read (Southwestern Section) Nov. 28, 1908. *Transactions of the American Mathematical Society*, vol. 10, No. 2, pp. 220–246 ; Apr., 1909.

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YOUNG, J. W. See VEBLEN, O.

INDEX.

Ames, L. D. A Simpler Proof of Lie's Theorem for Ordinary Differential Equations, 384.

Bill, E. G. The Construction of a Space Field of Extremals, 374.

—— See Reviews, under Scott.

Bliss, G. A. See Reviews, under Lebon.

Bolza, O. Heinrich Maschke: His Life and Work, 85.

—— Remarks Concerning the Second Variation for Isoperimetric Problems, 213.

Brown, E. W. See Reviews, under Annuaire, Jeans.

Buchanan, H. E. On Certain Determinants Connected with a Problem in Celestial Mechanics, 227.

Bussey, W. H. See Reviews, under Arnoux.

Cajori, F. See Reviews, under Voss.

Carmichael, R. D. Even Multiply Perfect Numbers of Five Different Prime Factors, 7.

—— Notes on the Simplex Theory of Numbers, 217.

Carver, W. B. Degenerate Pencils of Quadrics Connected with $\Gamma^{n+2}_{n+1,n}$ Configurations, 483.

Cole, F. N. Reports of Meetings of the American Mathematical Society: October Meeting, 157; Fifteenth Annual Meeting, 275; February Meeting, 323; April Meeting, 419.

Cowley, E. B. See Reviews, under Laurent.

Crathorne, A. R. See Reviews, under Scheffers.

Dickson, L. E. On the Representation of Numbers by Modular Forms, 338.

—— See Reviews, under Minkowski, Sylvester, Tannery.

Dodd, E. L. See Reviews, under Bois.

Dowling, L. W. See Reviews, under Czuber.

Ford, W. B. A Set of Criteria for the Summability of Divergent Series, 439.

Haskins, C. N. See Reviews, under Osgood.

Hawkes, H. E. See Reviews, under Enriques.

Hewes, L. I. See Reviews, under d'Ocagne.

Hoskins, L. M. General Algebraic Solutions in the Logic of Classes, 82.

—— A General Diagrammatic Method of Representing Propositions and Inference in the Logic of Classes, 84.

Hutchinson, J. I. See Reviews, under Durège, Picard.

Kasner, E. Tautochrones and Brachistochrones, 475.

—— See Reviews, under Loria, Schubert.

Kellogg, O. D. The Second Regular Meeting of the Southwestern Section, 207.

LONGLEY, W. R. See REVIEWS, under Andoyer, Jouguet.

LYTLE, E. B. See REVIEWS, under Slaught.

MCMAHON, J. On the Use of n-Fold Riemann Spaces in Applied Mathematics, 486.

MANNING, W. A. Reports of Meetings of the San Francisco Section of the American Mathematical Society: September Meeting, 105; February Meeting, 371.

MILLER, G. A. Answer to a Question Raised by Cayley as Regards a Property of Abstract Groups, 72.

—— On the Groups Generated by Two Operators Satisfying the Condition $s_1s_2 = s_2^{-2}s_1^{-2}$, 162.

—— The Sixtieth Meeting of the American Association for the Advancement of Science, 303.

—— See REVIEWS, under Müller, Royal Society.

MILLER, K. G. See WHITE, H. S.

MOORE, C. L. E. The Fourth International Congress of Mathematicians; Sectional Meetings, 8.

—— See REVIEWS, under Fabry.

MOORE, C. N. On Certain Constants Analogous to Fourier's Constants, 116.

MOULTON, F. R. See REVIEWS, under Poincaré.

OSGOOD, W. F. See REVIEWS, under Goursat.

OWENS, F. W. See REVIEWS, under Whitehead.

PIERPONT, J. See REVIEWS, under Hedrick, Hermite, Veblen.

RANUM, A. On Periodic Linear Substitutions Whose Coefficients are Integers, 4.

—— See REVIEWS, under Hilton.

RICHARDSON, R. G. D. The Cologne Meeting of the Deutsche Mathematiker-Vereinigung, 117.

RIETZ, H. L. See REVIEWS, under Blaschke.

SAUREL, P. On Fredholm's Equation, 445.

SCHWEITZER, A. R. On the Logical Basis of Grassmann's Extensive Algebra, 79.

—— Note on Enriques's Review of the Foundations of Geometry, 314.

—— See REVIEWS, under Hilbert.

SLAUGHT, H. E. The Fifteenth Summer Meeting of the American Mathematical Society, 57.

—— Reports of Meetings of the Chicago Section of the American Mathematical Society: Winter Meeting, 293; April Meeting, 425.

SMITH, D. E. See REVIEWS, under Ball, Cantor, Gray, Günther, Heath, White.

SMITH, E. M. Some Surfaces Having a Family of Helices as One Set of Lines of Curvature, 309.

SNYDER, V. Construction of Plane Curves of Given Order and Genus, Having Distinct Double Points, 1.

—— See Reviews, under Scheibner, Schoenflies, Sturm.

Tyler, H. W. See Reviews, under Chicago Symposium.

Underhill, A. L. The Second Variation of a Definite Integral, 379.

Westfall, W. D. A. Note on the Theorem of Generalized Fourier's Constants, 76.

—— The Solution of Boundary Problems of Linear Differential Equations of Odd Order, 223.

—— A Class of Functions Having a Peculiar Discontinuity, 225.

White, H. S. Bézout's Theory of Resultants and its Influence on Geometry, 325.

White, H. S., and Miller, K. G. Note on Lüroth's Type of Plane Quartic Curves, 347.

Wilczynski, E. J. Mathematical Appointments in Colleges and Universities, 492.

—— See Reviews, under Hesse.

Wilson, E. B. Note on Statistical Mechanics, 107.

—— See Reviews, under Fisher, Fleming, Grimsehl.

Wright, J. E. See Reviews, under Carslaw.

Young, J. W. See Reviews, under Bachmann.

Young, J. W. A. See Reviews, under Gutzmer, Klein.

Reviews.

Andoyer, H. Cours d'Astronomie. Seconde Partie : Astronomie pratique, W. R. Longley, 467.

Annuaire du Bureau des Longitudes pour l'An 1909, E. W. Brown, 364.

Arnoux, G. Arithmétique graphique. Les Espaces arithmétiques, leurs Transformations, W. H. Bussey, 255.

Aschkinass, E. See Fleming, J. A.

Bachmann, P. Grundlehren der neueren Zahlentheorie, J. W. Young, 463.

Baillaud, B. See Hermite, C.

Baker, H. F. See Sylvester, J. J.

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Bois, G. P. Tafeln unbestimmter Integrale, E. L. Dodd, 364.

Bourget, H. See Hermite, C.

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Fleischer, H. See Enriques, F.

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Freund, L. See Ball, W. W. R.

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Gundelfinger, S. See Hesse, O.

Günther, S. Geschichte der Mathematik, Theil 1, D. E. SMITH, 245.

Gutzmer, A. Die Tätigkeit der Unterrichtskommission der Gesellschaft Deutscher Naturforscher und Aerzte, J. W. A YOUNG, 261.

Heath, T. L. The Thirteen Books of Euclid's Elements translated from the Text of Heiberg with Introduction and Commentary, D. F. SMITH, 386.

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Hilbert, D Grundlagen der Geometrie (Dritte Auflage), A. R. SCHWEITZER, 510.

Hilton, H. An Introduction to the Theory of Groups of Finite Order, A. RANUM, 239.

Jeans, J. H. An Elementary Treatise on Theoretical Mechanics, E. W. BROWN, 165.

Jouguet, E. Lectures de Mécanique, Partie 1, W. R. LONGLEY, 466.

Klein, F., and Schimmack, R. Vorträge über den mathematischen Unterricht an den höheren Schulen, Theil 1, J. W. A. YOUNG, 511.

Laurent, H. La Géométrie analytique générale, E. B. COWLEY, 363.

Lebon, E. Table de Caractéristiques relatives a la Base 2310 des Facteurs premiers d'un Nombre inférieure à 30,030, G. A. Bliss, 139.

Lennes, N. J. See Veblen, O., and Slaught, H. E.

Loria, G. Il Passato ed il Presente delli principali Teorie geometriche (Terza Edizione), E. Kasner, 402.

Maurer, L, See Durège, H.

Minkowski, H. Diophantische Approximationen. Eine Einführung in die Zahlentheorie, L. E. Dickson, 251.

Müller, F. Führer durch die mathematische Literatur mit besonderer Berücksichtigung der historisch wichtigen Schriften, G. A. Miller, 403.

Ocagne, M. d'. Calcul graphique et Nomographie. Le Calcul simplifié. L. I. Hewes, 127.

Osgood, W. F. A First Course in the Differential and Integral Calculus, C. N. Haskins, 457.

Picard, E., and Simart, G. Théorie des Fonctions algébriques de deux Variables indépendantes, J. I. Hutchinson, 495.

Poincaré, H. Leçons de Mécanique céleste professées a la Sorbonne, F. R. Moulton, 258.

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Scheibner, W. Beiträge zur Theorie der linearen Transformationen als Einleitung in die algebraische Invariantentheorie, V. Snyder, 137.

Schoenflies, A. Einführung in die Hauptgesetze der zeichnerischen Darstellungsmethoden, V. Snyder, 362.

Schimmack, R. See Klein, F.

Schubert, H. Auslese aus meiner Unterrichts- und Vorlesungspraxis, E. Kasner, 400.

Scott, C. A. Cartesian Plane Geometry, Part 1: Analytical Conics, E. G. Bill, 507.

Serret, J. A. See Scheffers, G.

Simart, G. See Picard, E.

Slaught, H. E., and Lennes, N. J. High School Algebra, E. B. Lytle, 357.

Sturm, R. Die Lehre von den geometrischen Verwandtschaften, V. Snyder, Erster Band, 135 ; Zweiter Band, 252.

Sylvester, J. J. Collected Mathematical Papers (edited by H. F. Baker), Volumes 1–2, L. E. Dickson, 232.

Tannery, J. Manuscrits de Evariste Galois, L. E. Dickson, 249.

Veblen, O., and Lennes, N. J. Introduction to Infinitesimal Analysis. Functions of One Real Variable, J. Pierpont, 133.

Voss, A. Ueber das Wesen der Mathematik, F. Cajori, 405.

White, W. F. A Scrapbook of Elementary Mathematics, D. E. Smith, 190.

Whitehead, A. N. The Axioms of Descriptive Geometry, F. W. Owens, 465.

n, 145, 515.

530.

ublications, 51, 101, 151, 202, 269, 319, 368, 416, 471, 520.

es, 43, 95, 145, 198, 264, 316, 365, 410, 468, 515.

ers read before the Society and Subsequently Published, Eighteenth Annual List of, 523.

NOTES AND OTHER ITEMS.

ademies, Associations, Congresses, and Societies :

American Mathematical Society: Annual Meeting, 95, 145 ; Annual Register, 198 ; Chicago Section, 145, 264, 365 ; Chicago Symposium, 264, 515 ; Colloquium, 198, 468 ; Election of Officers, 276 ; New Members Admitted, 57, 157, 276, 323, 419 ; Presidential Address, 95, 145 ; Publishers, 515 ; Southwestern Section, 95 ; Summer Meeting, 198, 468 ; Transactions, 43, 96, 264, 323, 410, 419, 515.

Associations for the Advancement of Science: American, 265 ; Australasian, 147 ; British, 96, 468 ; French, 96 ; Italian, 146 ; Spanish, 317.

Associations of Teachers of Mathematics : American Federation, 145, 469, 515 ; International Commission, 265, 316, 411 ; Middle States, Syracuse Section, 265 ; Swiss, 199.

Astronomische Gesellschaft, 318 ; Belgian Academy, 44, 518 ; Berlin Academy, 46 ; Circolo Matematico di Palermo, 265, 365 ; Deutsche Mathematiker-Vereinigung, 316, 365, 516 ; Göttingen Academy, 44, 45 ; International Congress of Philosophy, 199 ; Italian Scientific Society, 98 ; Jablonowski Society, 518 ; Lombardy Royal Institute, 45 ; London Mathematical Society, 146, 199, 316, 410, 468 ; Mathesis, 44 ; National Academy, 470 ; Paris Academy, 265 ; Turin Academy, 317.

ks, Announcements of New, 46, 200, 264, 267, 411, 515.

ogues of Books, Models, etc., 46, 100, 147, 150, 268, 318, 319, 411, 471.

rates in Mathematics : American, 97 ; Paris, 318.

's Works, Publication of, 515.

Memorial, 318.

ls :

American Journal of Mathematics, 43, 96, 316, 410 ; American Mathetical Monthly, 411, 516 ; Annals of Mathematics, 43, 145, 265, 410 ; ansactions of the American Mathematical Society, 43, 96, 264, 323, 0, 419, 515.

and Communications Presented to the Society, Authors :

lardice, R. E., 371, 371.
nes, L. D , 58, 427.
ker, R. P., 426.
tes, W. H., 295, 427.
l, E. G., 277, 278.
chfeldt, H. F., 105.
s, G A., 58, 278, 294.
za, O., 58.
nke, W. C., 207.
hanan, H. E., 295, 295, 426.
k, T., 426.
michael, R. D., 157, 157, 277, 77, 419, 419.
Carver, W. B., 278.
Chessin, A. S., 420.
Coble, A. B., 278.
Conner, J. R., 278.
Craig, C. F., 278.
Davis, E. W., 207, 426
Dederick, L. S., 420.
Denton, W. W., 295.
Dickson, L. E , 58, 59, 59, 294, 295, 426, 427, 427.
Dowling, L. W., 59.
Dresden, A., 58.
Eiesland, J., 277.

Eisenhart, L. P., 420.
Fite, W. B., 59, 278.
Ford, W. B., 426, 426.
Frizell, A. B., 207, 208, 294, 295.
Gaba, M., 207.
Gillespie, D. C., 323.
Glenn, O. E., 58.
Glover, J. W., 294.
Granville, W. A., 278.
Griffin, F. L., 58, 277.
Hardy, J. G., 278.
Hartwell, G. W., 208.
Haskins, C. N., 59, 278, 426.
Hedrick, E. R., 59, 208.
Hewes, L. I., 59.
Hildebrandt, T. H., 427.
Hutchinson, J. I., 277.
Ingold, L., 58.
Irwin, F., 158.
Jackson, W. H., 420, 420.
Kasner, E., 59, 157, 278, 278, 323, 420.
Keyser, C. J., 278.
Landry, A. E., 158.
Leib, D. D., 278.
Lipke, J., 277.
Longley, W. R., 59, 277.
Lovett, E. O., 277.
Lunn, A. C., 294, 295, 295.
McDonald, J. H., 105.
McEwen, G. F., 371.
Maclagan-Wedderburn, J. H., 59.
MacMillan, W. D., 427, 427.
Manning, W. A., 371.
Mason, M., 58.
Miles, E. J., 294, 294.
Miller, G. A., 58, 59, 157, 207, 277, 426.
Moore, C. L. E., 277.
Moore, C. N., 59.
Moore, E. H., 295.
Morehead, J. C., 420, 42
Moreno, H. C., 105.
Morley, F., 277.
Moulton, F. R., 427, 427.
Neikirk, L. I., 427.
Newson, H. B., 59.
Noble, C. A., 105.
Osgood, W. F., 420.
Pell, A. J., 427.
Phillips, H. B., 278.
Pitcher, A., 427.
Ranum, A., 58, 58, 105, 278, 426.
Rietz, H. L., 427.
Roe, E. D., 420.
Roever, W. H., 207.
Saurel, P. L., 420.
Schweitzer, A. R., 295.
Sharpe, F. R., 58, 277.
Shaw, J. B., 58, 295.
Sinclair, M. E., 27 .
Snyder, V., 58, 58, 278, 323.
Stager, H. W., 371.
Study, E., 58.
Townsend, E. J., 59.
Underhill, A. L., 295.
Van der Vries, J. N., 207.
Wernicke, A. L. P., 207, 208.
Western, A. E., 426.
White, H. S., 277.
Whitney, A. W., 371.
Wilczynski, E. J., 59, 294, 4
Wilson, E. B., 59, 158.
Wright, J. E., 420.
Young, A. E., 426.
Young, J. W., 59, 295.

Personal Notes :

Almansi, E., 98 ; Appell, P., 96 ; Archibald, R. C., 150, 201 ;
C., 518.

Bagnera, G., 367 ; Bauer, G. A., 519 ; Bianchi, L., 266 ; Bierm
520 ; Biklé, C. E., 51 ; Bill, E. G., 97 ; Birkhoff, G. D., 319 ; Blis
100, 198, 468 ; Bôcher, M., 470, 471 ; Boggio, T., 49, 518 ; Bo
470 ; Bonola, R., 98 ; Börger, R. L., 97 ; Bourgeois, M., 99 ; Bou
P., 201 ; Brown, E. W., 265 ; Browne, J. J., 416 ; Buchanan,
520 ; Buhl, E., 367 ; Bullard, J. J., 100 ; Burgatti, P., 49, 415 ;
hardt, H., 49, 149 ; Butterfield, A. D., 100.

Cameron, J. F. 519 ; Carathéodory, C., 150, 518 ; Carrus, P.
Chambers, G. G., 97 ; Chase, F. L., 266 ; Chessin, A. S., 99 ; Ciso
201 ; Claussen, H., 46 ; Clements, G. R., 150 ; Cleveland, J. B
Cobb, C. W., 100 ; Conant, L. L., 100 ; Conwell, G. M., 51, 97 ;
E. B., 97 ; Craig, C. F., 97 ; Crathorne, A. R., 150.

Dalaker, H. H., 50 ; Darboux, G., 200 ; Darwin, F., 96 ; Dar
H., 200 ; Davisson, S. C., 50 ; Debierne, —, 266 ; DeFranchis,
Delassus, M., 319 ; Dickson, L. E., 411 ; Dirichlet, P. G. L.